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Mantas Dirma

Asymptotic Properties of Transformations of Heavy-Tailed Distributions

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Academic Supervisor – **Prof. Dr. Jonas Šiaulys** (Vilnius University, Natural Sciences, Mathematics – N 001).
Academic Consultant – **Prof. Dr. Ramūnas Garunkštis** (Vilnius University, Natural Sciences, Mathematics – N 001).

This doctoral dissertation will be defended in a public meeting of the Dissertation Defence Panel:
Chairman – **Prof. Dr. Darius Šiaučiūnas** (Vilnius University, Natural Sciences, Mathematics – N 001).
Members:
Prof. Dr. Paulius Drungilas (Vilnius University, Natural Sciences, Mathematics – N 001),
Assoc. Prof. Dr. Andrius Grigutis (Vilnius University, Natural Sciences, Mathematics – N 001),
Prof. Dr. Yuliya Mishura (Taras Shevchenko National University of Kyiv, Natural Sciences, Mathematics – N 001),
Prof. Dr. Olga Štikonienė (Vilnius University, Natural Sciences, Mathematics – N 001).

The dissertation shall be defended at a public meeting of the Dissertation Defence Panel at 3 p.m. on 11 September 2026 in Room 102 of the Faculty of Mathematics and Informatics, Vilnius University. Address: Naugarduko st. 24, Vilnius, Lithuania, tel. +370 5 219 3050; e-mail: mif@mif.vu.lt.

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Mantas Dirma

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Mokslinis vadovas – prof. dr. Jonas Šiaulys (Vilniaus universitetas, gamtos mokslai, matematika – N 001).
Mokslinis konsultantas – prof. dr. Ramūnas Garunkštis (Vilniaus universitetas, gamtos mokslai, matematika – N 001).

Gynimo taryba:
Pirmininkas – prof. dr. Darius Šiaučiūnas (Vilniaus universitetas, gamtos mokslai, matematika – N 001),
Nariai:
prof. dr. Paulius Drungilas (Vilniaus universitetas, gamtos mokslai, matematika – N 001),
doc. dr. Andrius Grigutis (Vilniaus universitetas, gamtos mokslai, matematika – N 001),
prof. dr. Yuliya Mishura (Kyjivo Taraso Ševčenkos nacionalinis universitetas, gamtos mokslai, matematika – N 001),
prof. dr. Olga Štikonienė (Vilniaus universitetas, gamtos mokslai, matematika – N 001).

Disertacija ginama viešame Gynimo tarybos posėdyje 2026 m. rugsėjo 11 d. 15 val. Vilniaus universiteto Matematikos ir informatikos fakulteto 102 auditorijoje. Adresas: Naugarduko g. 24, Vilnius, Lietuva, tel. +370 5 219 3050; el. paštas: mif@mif.vu.lt.

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Mantas Dirma

Contents

Abbreviations

r.v.	random variable
r.v.s	random variables
d.f.	distribution function
d.f.s	distribution functions
t.f.	tail distribution function
t.f.s	tail distribution functions
i.i.d.	independent and identically distributed (random variables)
pSQAI	pairwise strongly quasi-asymptotically independent
pTQAI	pairwise tail quasi-asymptotically independent
pAI	pairwise asymptotically independent
pQAI	pairwise quasi-asymptotically independent
FGM	Farlie-Gumbel-Morgenstern (copula)
AMH	Ali-Mikhail-Haq (copula)
VaR	Value-at-Risk (risk measure)
ES	Expected Shortfall (risk measure)
CVaR	Conditional Value-at-Risk (risk measure)
HG	Haezendonck-Goovaerts (risk measure)
MC	Monte Carlo (simulations method)

Notations

$\mathbb{R}$	Set of real numbers.
$\mathbb{R}_+$	Set of nonnegative real numbers, i.e., $\mathbb{R}_+ := [0, \infty)$.
$\overline{\mathbb{R}}$	Set of extended real numbers, i.e., $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$.
$\mathbb{N}$	Set of natural numbers, i.e., $\mathbb{N} := \{1, 2, 3, \ldots\}$.
$\mathbb{N}_0$	Set of nonnegative integers, i.e., $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.
F_ξ	Cumulative d.f. of a r.v. ξ, i.e., $F_\xi(x) := \mathbb{P}(\xi \leqslant x)$. We write F without a subscript when the corresponding r.v. is clear from the context.
$\overline{F}_\xi$	T.f. of a r.v. ξ, i.e., $\overline{F}_\xi(x) := \mathbb{P}(\xi > x)$. We write $\overline{F}$ when the corresponding r.v. is clear from the context.
$F_\xi * F_\xi$	Two-fold convolution of a d.f. F_ξ, i.e., $F_\xi * F_\xi(x) := \mathbb{P}(\xi_1 + \xi_2 \leqslant x)$, where ξ_1 and ξ_2 are independent copies of ξ.
C_ϑ	Copula with parameter ϑ. We write C when the specific parameter is not relevant.
c_ϑ	Joint density function of a copula C_ϑ.
$F^{\leftarrow}$	Generalised inverse of a d.f. F, i.e., $F^{\leftarrow}(q) := \inf\{x \in \mathbb{R} : F(x) \geqslant q\}$, $q \in [0, 1]$.
$\overline{F}^{\leftarrow}$	Generalised inverse of a t.f. $\overline{F}$, i.e., $\overline{F}^{\leftarrow}(p) := \inf\{x \in \mathbb{R} : \overline{F}(x) \leqslant p\}$, $p \in [0, 1]$.
$\mathscr{H}$	Class of heavy-tailed distributions.
$\mathscr{S}$	Class of subexponential distributions.
$\mathscr{L}$	Class of long-tailed distributions.
$\mathscr{D}$	Class of dominatedly varying distributions.

$\mathscr{C}$	Class of consistently varying distributions.
$\mathscr{ERV}_{\{\alpha,\beta\}}$	Class of extended regularly varying distributions with indices $0 \leqslant \alpha \leqslant \beta$.
$\mathscr{ERV}$	Class of extended regularly varying distributions
$\mathscr{R}_\alpha$	Class of regularly varying distributions with index $\alpha \geqslant 0$.
$\mathscr{R}_0$	Class of slowly varying distributions.
$\mathscr{R}$	Class of regularly varying distributions.
$\mathscr{P}_{\mathscr{D}}$	Class of positively decreasing distributions.
J_F^+	Upper Matuszewska index J_F^+ of a d.f. F.
L_F	L-index of a d.f. F.
$\Phi_{\{\mu,\sigma\}}$	Normal (Gaussian) distribution with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$.
Φ	Standard normal distribution, i.e., $\Phi_{\{0,1\}}$.
$\mathcal{U}([a,b])$	Uniform distribution on the interval $[a,b]$, $a < b$.
$\mathcal{P}a_{\{a,b\}}$	Pareto distribution with parameters $a, b > 0$.
$\mathcal{PP}_{\{a,b\}}$	Generalised Peter and Paul distribution with parameters $b > 1$, $a \in (0,\infty)$.
$\mathcal{CT}_p$	Cai-Tang distribution with parameter $p \in (0,1)$.
$\stackrel{d}{=}$	"Distributed as". For example, if a r.v. X has a standard normal distribution Φ, we write $X \stackrel{d}{=} \Phi$.
S_n^ξ	Sum of $n \in \mathbb{N}$ real-valued r.v.s $\xi_1, \ldots, \xi_n$, i.e., $S_n^\xi := \sum_{k=1}^n \xi_k$.
$S_n^{\theta\xi}$	Sum of $n \in \mathbb{N}$ real-valued r.v.s $\xi_1, \ldots, \xi_n$ weighted by nonnegative r.v.s $\theta_1, \ldots, \theta_n$, i.e., $S_n^{\theta\xi} := \sum_{k=1}^n \theta_k \xi_k$.
Π_n	Product of $n \in \mathbb{N}$ standard normal r.v.s $\xi_1, \ldots, \xi_n$, i.e., $\Pi_n := \prod_{k=1}^n \xi_k$.
$\widehat{\Pi}_n$	Product of $n \in \mathbb{N}$ normal r.v.s $\xi_1, \ldots, \xi_n$, i.e., $\widehat{\Pi}_n := \prod_{k=1}^n \xi_k$.
VaR_q	Value-at-Risk risk measure at level q.
ES_q	Expected Shortfall risk measure at level q.

CVaR_q	Conditional Value-at-Risk risk measure at level q.
HG_q	Haezendonck-Goovaerts risk measure at level q.
$\lfloor x \rfloor$	Floor function of a real number x, i.e., the greatest integer less than or equal than x.
$\lceil x \rceil$	Ceiling function of a real number x, i.e., the least integer greater or equal to x.
$\langle x \rangle$	Fractional part of a real number x, i.e., $\langle x \rangle := x - \lfloor x \rfloor$.
x^+	Positive part of a real number x, i.e., $x^+ := \max\{0, x\}$.
x^-	Negative part of a real number x, i.e., $x^- := \max\{0, -x\}$.
$\mathbb{I}_A$	Indicator function of a set A, i.e., $\mathbb{I}_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$

For two real functions f and g, we write

$f(x) = o(g(x))$	if	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

For two positive real functions f and g, we write

$f(x) = O(g(x)))$	if	$\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$ or, equivalently, if there exist $D_1, D_2 > 0$ such that $f(x) \leqslant D_1 g(x)$ for $x > D_2$.
$f(x) \underset{x\to\infty}{\sim} g(x)$	if	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.
$f(x) \underset{x\to\infty}{\lesssim} g(x)$	if	$\limsup_{x\to\infty} \frac{f(x)}{g(x)} \leqslant 1$.
$f(x) \asymp g(x)$	if	$f(x) = O(g(x))$ and $g(x) = O(f(x)))$.

1 Introduction

1.1 Research topic, problems and relevance

The heavy-tailed phenomenon, which describes the tail behaviour of a distribution, roughly speaking, means that large values under a given distribution are assigned relatively high probabilities, or more precisely that a t.f. decays significantly slower than any exponential t.f. (see also Lemma 2.1 below). Contrary to the more classical and widespread theory in probability and statistics, which is centred on lighter tailed distributions, Gaussianity, central moments, and central limit theorem, heavy-tailed behaviour is governed by large values in the tails [92]. As a result, when dealing with heavy-tailed distributions, asymptotic tail behaviour becomes a vital characteristic.

Over the years, heavy-tailed distributions have received considerable attention in a wide range of fields, including applied probability theory (risk theory, extreme value theory), economics and finance, insurance, information networks, and environmental sciences, and are often used to model relatively rare but extreme events that classical distributions tend to underestimate. Due to the complex nature of the systems in which they are applied, various transformations of heavy-tailed distributions are often considered – for instance, sums (see [56, 25, 132, 26] among many others) and products (see [30, 31, 68], [74, Chapter 5]) of r.v.s distributed in this way or certain functionals, such as generalised moments (see [76, 105]) or risk measures (as in [108, 80, 109]).

1.1.1 Products of normal random variables: from light to heavy

The normal (Gaussian) distribution is often taken as a counterexample to heavy-tailed distributions. It is arguably the most widely used distribution in probability and statistics and has well-known properties. In contrast, the distributional behaviour of the product of two or more normal r.v.s to this day remains the subject of active research; see [82, 34, 54, 55] and the references therein. In many of the articles, the exact expressions of the density of product of two normal r.v.s have been explored (see Section 3.1). However, the formulas obtained involve integrals or infinite series, which hinder their direct applicability and generalisation to the case of more multipliers. Consequently, asymptotic properties play an important role in approximating the distributional behaviour. This is even more significant, as, surprisingly, products of independent normal r.v.s have heavy tails when three or more multipliers are involved (see Lemma 3.1). More formally, let $n \in \mathbb{N}$ be given and consider a collection of independent standard normal r.v.s $\{\xi_1, \ldots, \xi_n\}$. Denote the product of such r.v.s by

$$\Pi_n := \prod_{k=1}^{n} \xi_n.$$

Then it can be shown that Π_n is heavy-tailed if $n \geqslant 3$ and light-tailed otherwise. In a relatively recent study [6], asymptotic formulas were obtained, that could be applied to the tail distribution

$$\mathbb{P}(\Pi_n > x) \qquad \text{as} \qquad x \to \infty.$$

The formulas are given in an asymptotically exact form; however, since no error terms are provided, the asymptotic convergence rate is not clear.

1.1.2 Random(-ly weighted) sums of heavy-tailed random variables

Random outcomes in a system can often be expressed as a sum of r.v.s. This occurs, for instance, when modelling total claims in an insurance portfolio, where each claim amount is an individual summand, or in financial contexts, where total losses of a portfolio are comprised of losses of underlying assets, such as stocks or bonds. In cases where the contribution of an individual component in a system is stochastic, the aforementioned sums may well be generalised to a weighted setting where each summand is multiplied by a random weight.

To formalise the above discussion, let $n \in \mathbb{N}$ be fixed and $\{\xi_1, \ldots, \xi_n\}$

be a collection of real-valued and possibly dependent heavy-tailed r.v.s, also called *primary* r.v.s. Additionally, suppose that $\{\theta_1, \dots, \theta_n\}$ is a collection of nonnegative, nondegenerate at zero r.v.s, called *random weights.* Consider sums of the primary r.v.s

$$S_n^{\xi} := \sum_{k=1}^{n} \xi_k = \xi_1 + \ldots + \xi_n \tag{1.1}$$

and that of the randomly weighted summands

$$S_n^{\theta\xi} := \sum_{k=1}^{n} \theta_k \xi_k = \theta_1 \xi_1 + \ldots + \theta_n \xi_n. \tag{1.2}$$

During recent years, the asymptotics of sums (1.1) and (1.2) have been studied extensively in the literature of applied probability in the context of risk and extreme value theories (for a more detailed overview, see Section 4.1). In particular, many studies focus on the asymptotic behaviour of the tail probabilities

$$\mathbb{P}\left(S_n^{\xi} > x\right) \qquad \text{and} \qquad \mathbb{P}\left(S_n^{\theta\xi} > x\right), \qquad \text{as} \qquad x \to \infty,$$

expressing them as sums of tail probabilities of the individual summands, i.e., $\sum_{k=1}^{n} \mathbb{P}(\xi_k > x)$ and $\sum_{k=1}^{n} \mathbb{P}(\theta_k \xi_k > x)$ respectively (see [25, 56, 26, 59, 62, 76, 107, 113, 114, 132] and the references therein). In addition, the asymptotic behaviour of the tail expectations

$$\mathbb{E}\left(S_n^{\xi}\, \mathbb{I}_{\{S_n^{\xi} > x\}}\right) \qquad \text{and} \qquad \mathbb{E}\left(S_n^{\theta\xi}\, \mathbb{I}_{\{S_n^{\theta\xi} > x\}}\right), \qquad \text{as} \qquad x \to \infty,$$

has been studied; however, the literature remains relatively scarce (see, e.g., [62, 76, 128, 105]). More generally, the problem could be extended to the case of generalised tail moments

$$\mathbb{E}\left(\varphi(S_n^{\xi})\, \mathbb{I}_{\{S_n^{\xi} > x\}}\right) \qquad \text{and} \qquad \mathbb{E}\left(\varphi(S_n^{\theta\xi})\, \mathbb{I}_{\{S_n^{\theta\xi} > x\}}\right), \qquad \text{as} \qquad x \to \infty,$$

where $\varphi \colon \mathbb{R} \to \mathbb{R}$ is a specific measurable transformation.

It is worth mentioning that the sums (1.1) and (1.2) frequently arise in applications in finance and insurance. In the following, we give three more specific use cases:

- The first is related to the discrete-time risk model in which a primary r.v. ξ_k, $k \in \mathbb{N}$, represents the net loss (total claim amount minus the total premium income) of an insurance company in period k, while a random weight θ_k corresponds to the stochastic discount factor used to discount insurer's cash flows from period k to the present time. In this setting, $S_n^{\theta\xi}$ denotes the present value of the total net loss of the company accumulated over the first n periods. For more details, see [85, 106, 59, 114, 51, 110, 116] among others.

- The second insurance-related example is based on the individual risk model (see [112, 37], [38, Chapter 4]). Assume that an insurance company has a portfolio of n policies with independent random claims $X_1, \ldots, X_n$. Since there is a possibility that there is no claim on a policy, each random claim X_k is decomposed into the product $X_k = \theta_k \xi_k$, $k = 1, \ldots, n$, where θ_k is an indicator Bernoully r.v., which is equal to one if the k-th claim occurs and zero otherwise, and ξ_k is the claim size conditional on a claim event. Then the sum $S_n^{\theta\xi}$ corresponds to the total amount of claims incurred in the insurance portfolio.

- The third use case concerns the construction of an investment portfolio and capital allocation (see [110]). Assume that a portfolio is formed from n distinct financial assets. In such a setting, r.v. ξ_k, $k \in \{1, \ldots, n\}$, denotes the size of future losses incurred from the k-th instrument in the portfolio, while the corresponding random weight θ_k admits several interpretations. It could be treated as a stochastic discount factor or as a weight, corresponding to the k-th instrument share in the portfolio. The randomly weighted sum $S_n^{\theta\xi}$ then either denotes the present value of the total loss of the portfolio or the total weighted loss of the portfolio.

1.1.3 Application to risk measures for aggregate risk

Financial and insurance activities are closely associated with the estimation of tail events. One of the main tools that allow to assess the risk of such events are risk measures – transformations (or, more precisely, functionals) which assign to each financial position a numerical (monetary) value representing its risk. On the one hand, they serve as a method to assess the amount of additional capital required to safeguard against potential losses. On the other hand, they might be interpreted as premium principles in an insurance context, which determine the premium an insurer should charge. As risk management depends on the level of risk aversion, most risk measures are parametrised by a confidence level $q \in (0, 1)$ that determines the degree of conservativeness. Generally, the higher the confidence level, the more conservative the assessment of the risk of a position.

The harsh lessons learnt from crises and the increasing complexity in financial markets have exposed the limitations of certain risk measures. In particular, widely used measures may fail to adequately capture tail risk, lack desirable coherence properties, or have limited flexibility to model risk preferences. For instance, a well-known Value-at-Risk (VaR) measure (see Definition 2.19) in general fails to obey the diversification principle and, due to its very definition, does not account for the severity of potential tail losses beyond a selected threshold (for more, see Section 2.4 below). After the Global Financial Crisis some researchers and regulators even argued that VaR was one of the contributing factors to the systemic failures (see, for instance, the discussion in [101]).

The aforementioned challenges have motivated the search for more robust and theoretically grounded alternatives. One of them is the Expected Shortfall (ES) risk measure (see Definition 2.20) which, contrary to VaR, is a coherent risk measure [9]. Another coherent and more general alternative is the Haezendonck–Goovaerts (HG) risk measure (see Definition 2.24), which depends on a Young function φ that governs its behaviour [15]. The flexibility to choose a preferred Young function gives more control over the degree to which extreme tail losses influence risk assessment. Interestingly, the HG risk measure reduces to the ES if a linear Young function is selected.

It is important to note that, despite being more theoretically grounded and flexible than some of the alternatives, the HG risk measure, due to its very definition involving generalised quantiles, might be less straightforward to implement in practice. In fact, as mentioned in [108], the analytical expression of the HG risk measure is, in general, not attainable, although for certain Young functions, such as power functions, more manageable expressions were obtained (see [108] and Section 5.1). For these reasons, statistical inference (see [119, 78, 134] and the references therein) and asymptotic formulas (see [80, 108, 109]), when the confidence level q tends to unit, are often considered in the literature. It should be noted that when the risk being measured is an aggregate of potentially dependent individual risks (represented by S_n^{ξ} and $S_n^{\theta\xi}$), the estimation often becomes even more complex as the distribution of the sum of r.v.s generally has to be estimated.

1.2 Main goals of the research

The main goals of this Doctoral dissertation focus on the asymptotic properties of three types of transformations of heavy-tailed r.v.s discussed in the previous section and can be summarised as follows:

- Obtain more accurate asymptotic formulas for the tail probability of the product Π_n of n independent standard normal r.v.s. In particular, the exact asymptotic equivalence

$$\mathbb{P}\left(\Pi_n > x\right) \underset{x\to\infty}{\sim} W_n(x),$$

  where W_n is a real function dependent on x and n, which could be derived from the results in [6]. We intend to improve this asymptotic formula by uncovering error term(s), which would give more insight into the decay rate of the tail probability. The results could then be also generalised to the products $\widehat{\Pi}_n$ of n zero-mean normal r.v.s with arbitrary variances.

- Derive asymptotic bounds for the generalised tail moments of randomly weighted sums $S_n^{\theta\xi}$ with heavy-tailed and not necessarily independent primary r.v.s under a transformation from a certain class of transformations that would improve and generalise the previous results. More specifically, our objective is to define a class $\mathcal{K}$ of measurable functions such that, if $\varphi \in \mathcal{K}$, then the tail moment of $\varphi(S_n^{\theta\xi})$ can be asymptotically bounded by the sums of tail moments of transformed individual summands $\varphi(\theta_k\xi_k)$, $k = 1, \ldots, n$, i.e.,

$$\mathbb{E}\left(\varphi(S_n^{\theta\xi})\mathbb{I}_{\{S_n^{\theta\xi}>x\}}\right) \underset{x\to\infty}{\gtrsim} \sum_{k=1}^{n} C_{\xi_k}\mathbb{E}(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta\xi>x\}}),$$

$$\mathbb{E}\left(\varphi(S_n^{\theta\xi})\mathbb{I}_{\{S_n^{\theta\xi}>x\}}\right) \underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{C_{\xi_k}}\mathbb{E}(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta\xi>x\}}),$$

where C_{ξ_k} is a special correcting constant depending on the r.v. ξ_k.

- Establish asymptotic formulas of the Haezendonck-Goovaerts risk measure specified by the power Young function $\varphi(t) = t^{\varkappa}$ for the sums S_n^{ξ} and $S_n^{\theta\xi}$ with heavy-tailed and not necessarily independent random summands, when the confidence level of the measure tends to 1. Relying on the exact formulas obtained in [108] we seek to obtain expressions that would allow us to estimate the value of the HG measure for such sums by a functional depending on the marginal distributions of individual risks ξ_k or $\theta_k\xi_k$, $k = 1, \ldots, n$, without the need to estimate the distributions of S_n^{ξ} or $S_n^{\theta\xi}$.

1.3 Methodology

To obtain the main research goals, we predominantly use the classical results of probability theory and real analysis. Additionally, we rely on the probabilistic extreme value theory techniques to explore the tail asymptotics of heavy-tailed r.v.s, utilise properties of quantile functions when dealing with risk measures, and employ copula theory to model dependence between r.v.s.

To empirically validate and illustrate the obtained theoretical results, we present examples based on Monte Carlo (MC) simulations with selected distributions and specific copula-based dependence structures. However, we note that the derivations in the proofs of the main results are purely analytical and do not involve any numerical approximations.

The MC simulations mentioned above were performed using the statistical software **R** (v4.2.3) [90], the figures presented in this Doctoral Dissertation were also generated in **R** using the packages *ggplot2*[122] and *tikzDevice*[97].

1.4 Publications

The main results presented in this Doctoral Dissertation have been published in the following scientific articles. All of them appear in the journals indexed by ISI Web of Science.

- M. Dirma, S. Paukštys, and J. Šiaulys. Tails of the moments for sums with dominatedly varying random summands. *Mathematics*, 9(8):824, 2021.

- R. Leipus, J. Šiaulys, M. Dirma, and R. Zovė. On the distribution-tail behaviour of the product of normal random variables. *Journal of Inequalities and Applications*, 2023(1):32, 2023.

- M. Dirma, N. Nakliuda, and J. Šiaulys. Generalized moments of sums with heavy-tailed random summands. *Lithuanian Mathematical Journal*, 63(3):254-271, 2023.

- J. Šiaulys, M. Dirma, N. Nakliuda, and L. Zanardelli. Asymptotic Formulas for the Haezendonck–Goovaerts Risk Measure of Sums with Consistently Varying Increments. *Axioms*, 15(1):20, 2026.

1.5 Conferences and seminars

The results obtained were presented in the following conferences and seminars:

- M. Dirma and N. Nakliuda, *Asimptotinės formulės sunkiauodegių skirstinių sumų apibendrintiems momentams.* Finansų ir draudimo matematikos seminaras, 13 December 2022, Vilnius.

- M. Dirma, *Asimptotinės formulės sunkiauodegių skirstinių sumų apibendrintiems momentams.* 11-asis Lietuvos jaunųjų matematikų susitikimas, 30 December 2022, Kaunas.

- M. Dirma, *Apibendrinti atsitiktinių sumų su sunkiauodegiais dėmenimis momentai.* 64-oji Lietuvos Matematikų Draugijos konferencija, 21–22 June 2023, Vilnius.

- M. Dirma, *Normaliųjų atsitiktinių dydžių sandaugų skirstinio uodegos asimptotinis elgesys*, 65-oji Lietuvos Matematikų Draugijos konferencija, 27–28 June 2024, Šiauliai.

- M. Dirma, *On the Distribution-Tail Behaviour of the Product of Normal Random Variables*, International Conference on Probability Theory and Number Theory 2024, 16–20 September 2024, Palanga.

- M. Dirma and N. Nakliuda, *Haezendonck-Goovaerts rizikos mato asimptotika skirstinių su nuosaikiai kintančiomis uodegomis sumoms*, 66-oji Lietuvos Matematikų Draugijos konferencija, 26–27 June 2025, Klaipėda.

- M. Dirma, *On the asymptotic behaviour of the Haezendonck-Goovaerts risk measure for sums with consistently varying increments*, International Vilnius Conference on Statistics and Its Applications, 27–29 August 2025, Vilnius.

1.6 Structure of the dissertation

The rest of the dissertation is structured as follows. Chapter 2 introduces the key concepts and some general results that are used throughout the thesis. In particular, Section 2.1 reviews the main classes of heavy-tailed and related distributions, Section 2.2 presents dependence structures that are similar to asymptotic independence and the main facts from copula theory used to model dependence. Section 2.3 defines the concept of a generalised inverse function and presents its properties, Section 2.4 introduces risk measures and provides several examples of them, including the Haezendonck-Goovaerts risk measure. The chapter concludes with Section 2.5, which presents a couple of useful general results.

The following three chapters focus on the main research topics of the dissertation. In Chapter 3 the tail probability asymptotics of the product of independent zero-mean normal random variables is investigated. Chapter 4 explores the asymptotic properties of the random sums with dominatedly varying random summands under certain transformations. Chapter 5 presents the asymptotics of the Haezendonck-Goovaerts risk measure for the sums with consistently varying risks. Each of the three chapters begins with an overview of related results in the literature and a broader outline of the research problem. Then, the main results are listed, with their proofs presented in a separate section. At the end of each chapter, several examples are provided, which illustrate the validity of the results via the MC simulations.

Finally, Chapter 6 concludes the results obtained and presents possible directions for future research.

2 Preliminaries

2.1 Classes of distributions

In this section, we review the main classes of distributions used in this dissertation. Before introducing the fundamental class of heavy-tailed distributions, we first present the notion of the support of a distribution. The *support* of a d.f. F is defined as

$$\text{supp}\{F\} := \{x \colon F(x+\delta) - F(x-\delta) > 0 \quad \text{for all} \quad \delta > 0\}\,.$$

We say that a d.f. F is supported on $\mathbb{R}$ if $\text{supp}\{F\} \subseteq \mathbb{R}$.

Definition 2.1. *A d.f. F supported on $\mathbb{R}$ is said to be* heavy-tailed*, written as $F \in \mathscr{H}$, if, for every $\delta > 0$, it holds that*

$$\int_{\mathbb{R}} \mathrm{e}^{\delta x} \mathrm{d}F(x) = \infty.$$

Note that, by the alternative expectation formula (see Lemma 2.5), we have

$$\int_{[0,\infty)} \mathrm{e}^{\delta x} \mathrm{d}F(x) = 1 + \delta \int_0^\infty \mathrm{e}^{\delta x} \overline{F}(x) \mathrm{d}x,$$

implying that

$$F \in \mathscr{H} \implies \overline{F}(x) > 0,\, x \in \mathbb{R}.$$

In other words, Definition 2.1 says that a real-valued random variable ξ is heavy-tailed if its moment generating function $M_\xi(\delta) := \mathbb{E}(\mathrm{e}^{\delta\xi}) = \infty$ for all $\delta > 0$. To check whether a distribution belongs to the class $\mathscr{H}$, one can also use the lemma given below.

Lemma 2.1. *A d.f. F is heavy-tailed if and only if for every $\delta > 0$,*

$$\limsup_{x\to\infty} \frac{\overline{F}(x)}{\mathrm{e}^{-\delta x}} = \infty.$$

For the proof of Lemma 2.1, see [83, Lemma 1.2]. If a distribution is not heavy-tailed, we say that it is *light-tailed.*

Example 2.1 (Normal distribution)**.** *The normal distribution with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$, written as $\Phi_{\{\mu,\sigma\}}$, defined by a density function*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad x \in \mathbb{R},$$

and a d.f.

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \mathrm{d}x$$

is light-tailed. It is a well-known fact, that the moment generating function of $\Phi_{\{\mu,\sigma\}}$ is

$$M(\delta) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\infty}^{\infty} \exp\left\{\delta x - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \mathrm{d}x = \exp\left\{\delta\mu + \frac{(\sigma\delta)^2}{2}\right\}$$

for all $\delta > 0$. We denote by $\Phi := \Phi_{\{0,1\}}$ the case of standard normal distribution.

Example 2.2 (Exponential distribution)**.** *The exponential d.f. with parameter $\lambda > 0$, defined by*

$$F(x) = \left(1 - \mathrm{e}^{-\lambda x}\right) \mathbb{I}_{[0,\infty)}(x),$$

is light-tailed. Indeed, by Lemma 2.1 with $\delta = \lambda$, we get

$$\limsup_{x\to\infty} \frac{\overline{F}(x)}{\mathrm{e}^{-\delta x}} = \limsup_{x\to\infty} \frac{\mathrm{e}^{-\lambda x}}{\mathrm{e}^{-\delta x}} = 1.$$

In what follows, we present some of the most commonly encountered subclasses of heavy-tailed distributions, also referred to as regularity classes. We begin with arguably the most thoroughly studied class of distributions listed in this subsection, namely, the class of regularly varying functions, first formalised by Karamata in [64].

Definition 2.2. *A d.f.* F *supported on* $\mathbb{R}$ *is said to be* regularly varying *with index* $\alpha \geqslant 0$*, written as* $F \in \mathscr{R}_\alpha$*, if for all* $y > 0$*, it holds that*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}.$$

$\mathscr{R}_0$ *is also called a class of* slowly varying *distributions. We denote by* $\mathscr{R} := \bigcup_{\alpha\geqslant 0} \mathscr{R}_\alpha$ *the class of regularly varying distributions.*

As noted in [91], the class $\mathscr{R}$, broadly speaking, consists of functions which "behave asymptotically like power functions". The class of regularly varying functions has been widely used in probability theory and real analysis (see [32, 96, 63, 104, 75, 89] to name a few). For a systematic treatment of regularly varying functions, see also [16].

Example 2.3 (Pareto distribution)**.** *The Pareto d.f. with parameters* $a, b > 0$ *(written as* $\mathcal{P}a_{\{a,b\}}$*), defined by*

$$F(x) = \left(1 - \left(\frac{b}{b+x}\right)^a\right) \mathbb{I}_{[0,\infty)}(x), \tag{2.1}$$

is regularly varying with index a*, i.e.,* $F \in \mathscr{R}_a$*. Observe, that for all* $y > 0$*,*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \lim_{x\to\infty} \left(\frac{b+x}{b+xy}\right)^a = y^{-a}.$$

A related extension is the so-called class of extended regularly varying distributions, which generalises the regularly varying functions.

Definition 2.3. *A d.f.* F *supported on* $\mathbb{R}$ *is said to be* extended regularly varying *with indices* $0 \leqslant \alpha \leqslant \beta$*, written as* $F \in \mathscr{ERV}_{\{\alpha,\beta\}}$*, if for all* $y > 1$*, it holds that*

$$y^{-\beta} \leqslant \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant y^{-\alpha},$$

or, equivalently, for all $0 < y \leqslant 1$*,*

$$y^{-\alpha} \leqslant \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant y^{-\beta}.$$

We denote by $\mathscr{ERV} := \bigcup_{0 \leqslant \alpha \leqslant \beta} \mathscr{ERV}_{\{\alpha,\beta\}}$ *the class of extended regularly varying distributions.*

It can be easily seen that if $\alpha = \beta > 0$, then $\mathscr{ERV}_{\{\alpha,\beta\}} = \mathscr{R}_\alpha$. The properties of functions from $\mathscr{ERV}$ were studied in detail in [29, 67, 24, 121] (see also the references therein).

Next, we define the class of consistently varying distributions, which was introduced in [29] as a generalisation of the class $\mathscr{R}$ and was initially referred to as *intermediate regular variation.*

Definition 2.4. *A d.f. F supported on $\mathbb{R}$ is said to be* consistently varying*, written as $F \in \mathscr{C}$, if*

$$\lim_{y \uparrow 1} \limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

The class of consistently variation has attracted great interest and has been examined in depth (see, for instance, [31, 25, 23, 65, 5, 18, 117] and the references therein).

Example 2.4 (Cai-Tang distribution)**.** *An example of a consistently varying distribution was given in [23]. Let Y and N be independent r.v.s such that $Y \stackrel{d}{=} \mathcal{U}([0,1])$ and N is a geometric r.v. with parameter $p \in (0,1)$, i.e., $\mathbb{P}(N = k) = (1-p)p^k$ for $k = 0, 1, \ldots$ Then a r.v. X, defined by*

$$X = (1+Y)2^N, \tag{2.2}$$

belongs to the class $\mathscr{C}$ but not to the $\mathscr{R}$ (for the proof, see [23]). It can be shown that the r.v. X defined in (2.2) *has the following distribution function:*

$$\begin{aligned} F(x) = (1-p)\Bigg(&\frac{1 - p^{\lfloor \log_2 x \rfloor}}{1-p} \\ &+ \left(\frac{x}{2^{\lfloor \log_2 x \rfloor}} - 1\right) p^{\lfloor \log_2 x \rfloor}\Bigg) \mathbb{I}_{[1,\infty)}(x). \end{aligned} \tag{2.3}$$

We denote this distribution by $\mathcal{CT}_p$ and refer to it as the Cai-Tang distribution after the authors who applied it in [23]. Indeed, observe that

$$F(x) = \mathbb{P}\left((1+Y)2^N \leqslant x\right) = \sum_{k=0}^{\infty} \mathbb{P}\left((1+Y)2^k \leqslant x\right) \mathbb{P}(N = k)$$

$$= (1-p)\sum_{k=0}^{\infty} \mathbb{P}\left(Y \leqslant \frac{x}{2^k} - 1\right) p^k.$$

Clearly, $F(x) = 0$ *for* $x < 1$*. Now consider the case* $x \geqslant 1$*. Since* $Y \overset{d}{=} \mathcal{U}([0,1])$*, we get*

$$\mathbb{P}\left(Y \leqslant \frac{x}{2^k} - 1\right) = \mathbb{I}_{[2^{k+1},\infty)}(x) + \left(\frac{x}{2^k} - 1\right) \mathbb{I}_{[2^k,2^{k+1})}(x),$$

and therefore

$$\begin{aligned}
\sum_{k=0}^{\infty} \mathbb{P}\left(Y \leqslant \frac{x}{2^k} - 1\right) p^k &= \mathbb{I}_{[2,\infty)}(x) \left(\sum_{k=0}^{\lfloor \log_2 x \rfloor - 1} p^k\right) \\
&\qquad + \left(\frac{x}{2^{\lfloor \log_2 x \rfloor}} - 1\right) p^{\lfloor \log_2 x \rfloor} \mathbb{I}_{[1,\infty)}(x) \\
&= \frac{1 - p^{\lfloor \log_2 x \rfloor}}{1-p} \mathbb{I}_{[2,\infty)}(x) + \left(\frac{x}{2^{\lfloor \log_2 x \rfloor}} - 1\right) p^{\lfloor \log_2 x \rfloor} \mathbb{I}_{[1,\infty)}(x) \\
&= \left(\frac{1 - p^{\lfloor \log_2 x \rfloor}}{1-p} + \left(\frac{x}{2^{\lfloor \log_2 x \rfloor}} - 1\right) p^{\lfloor \log_2 x \rfloor}\right) \mathbb{I}_{[1,\infty)}(x),
\end{aligned}$$

implying (2.3).

The following broader class of heavy-tailed distributions, characterised by dominatedly varying tails, was first introduced in [47] and later extensively studied in [57, 106, 113, 114, 56, 130, 132, 26] among others.

Definition 2.5. *A d.f.* F *supported on* $\mathbb{R}$ *is said to be* dominatedly varying*, written as* $F \in \mathscr{D}$*, if for any fixed* $y \in (0,1)$*, it holds that*

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

Example 2.5 (Peter and Paul distribution)**.** *As noted in [57], the Peter and Paul distribution is an example of a dominatedly varying distribution. We say that a r.v.* X *is distributed according to the generalised Peter and Paul distribution with parameters* $b > 1$*,* $a \in (0,\infty)$ *(written as* $\mathcal{PP}_{\{a,b\}}$*), if its d.f. is defined by*

$$F(x) = (b^a - 1) \sum_{k \geqslant 1, b^k \leqslant x} b^{-ak}, \tag{2.4}$$

or, equivalently, if its t.f. is

$$\overline{F}(x) = (b^a - 1) \sum_{k \geqslant 1, b^k > x} b^{-ak}. \tag{2.5}$$

Since for $x \geqslant 1$,

$$\begin{aligned}\overline{F}(x) &= (b^a - 1)\left(\frac{1}{b^a - 1} - \sum_{k \geqslant 1, b^k \leqslant x} b^{-ak}\right)\\ &= (b^a - 1)\left(\frac{1}{b^a - 1} - \frac{\left(1 - \left(b^{-a}\right)^{\lfloor \log_b x \rfloor}\right)}{b^a - 1}\right) = \left(b^{-a}\right)^{\lfloor \log_b x \rfloor},\end{aligned}$$

we get

$$\begin{aligned}\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} &= \limsup_{x\to\infty} \left(b^{-a}\right)^{\lfloor \log_b x + \log_b y \rfloor - \lfloor \log_b x \rfloor}\\ &\leqslant \left(b^{-a}\right)^{\lfloor \log_b y \rfloor - 1}\end{aligned}$$

for all $y \in (0, 1)$, implying $X \in \mathscr{D}$.

We now introduce two extensively studied subclasses of $\mathscr{H}$, introduced by Chistyakov in [27] in the context of branching processes. The first is the class of subexponential distributions. Although this class was originally defined for distributions supported on a positive half-line of $\mathbb{R}$, we use an extended definition that also applies to the distributions supported on the whole $\mathbb{R}$ (see [50, Section 3.2]).

Definition 2.6. *A d.f. F supported on $\mathbb{R}$ is said to be* subexponential*, written as $F \in \mathscr{S}$, if for the d.f. $F_+(x) := F(x)\mathbb{I}_{[0,\infty)}(x)$,*

$$\lim_{x\to\infty} \frac{\overline{F_+ * F_+}(x)}{\overline{F}_+(x)} = 2.$$

The class of subexponential distributions has been further studied systematically in [28, 111, 57, 66, 107, 106, 56, 115, 129, 110] among a number of others. The second closely related and larger class is the class of long tailed distributions, considered in [42, 30, 66, 23, 125, 49] (see also the references therein).

Definition 2.7. *A d.f. F supported on $\mathbb{R}$ is said to be* long tailed*, written* *as* $F \in \mathscr{L}$, *if for all* $y > 0$,

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1.$$

For an in-depth overview of subexponential and long tailed distributions, also refer to Chapters 2 and 3 of the monograph by Foss et al. [50].

The interrelationships of the classes of heavy-tailed distributions can be expressed as follows (see, e.g. [57, 16, 23]):

$$\begin{gathered} \mathscr{R} \subset \mathscr{ERV} \subset \mathscr{C} \subset \mathscr{D} \cap \mathscr{L} \subset \mathscr{S} \subset \mathscr{L} \subset \mathscr{H}; \\ \mathscr{D} \subset \mathscr{H}; \qquad \mathscr{D} \setminus \mathscr{L} \neq \emptyset. \end{gathered} \tag{2.6}$$

A schematic illustration of (2.6) is shown in Figure 2.1.

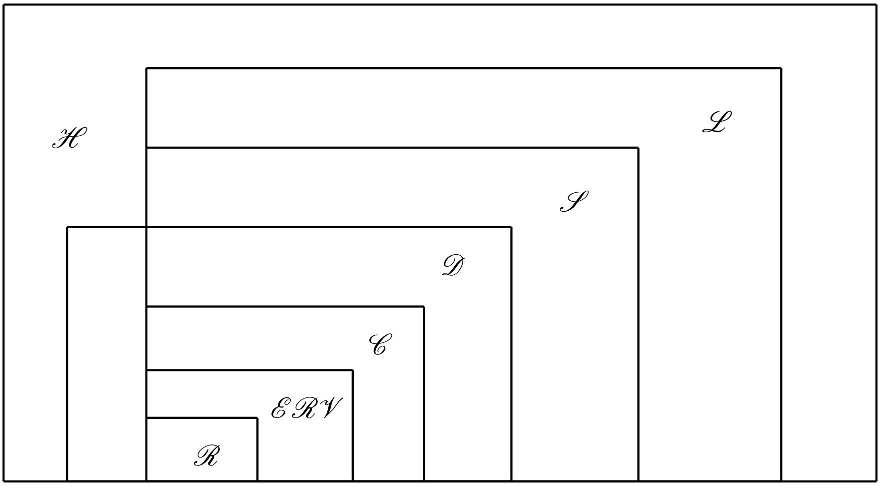

Figure 2.1: Classes of heavy-tailed distributions (each class is represented by a rectangle).

It is worth noting that all the inclusions in (2.6) are proper. Cline and Samorodnitsky in [31] provided two examples of distributions which prove the strictness of the inclusions $\mathscr{R} \subset \mathscr{ERV} \subset \mathscr{C}$. In particular, a d.f.

$$F(x) = \left(1 - \exp\left\{-\lfloor \log(x) \rfloor + (\log(x) - \lfloor \log(x) \rfloor)^2\right\}\right) \mathbb{I}_{[1,\infty)}(x) \tag{2.7}$$

is in $\mathscr{ERV} \setminus \mathscr{R}$, while a d.f.

$$F(x) = \Big(1 - \exp\Big\{-\lfloor \log(x) \rfloor + (\log(x) - \lfloor \log(x) \rfloor)^{1/2}\Big\}\Big)\, \mathbb{I}_{[1,\infty)}(x) \quad (2.8)$$

belongs to the class $\mathscr{C} \setminus \mathscr{ERV}$. The Peter and Paul distribution given in Example 2.5 belongs to the class $\mathscr{D} \setminus \mathscr{L}$ [57]. The Weibull distribution

$$F(x) = \Big(1 - \exp\Big\{x^{1/2}\Big\}\Big)\, \mathbb{I}_{[0,\infty)}(x) \quad (2.9)$$

is an example of a subexponential distribution that is not dominatedly varying [57]. The examples of distributions that belong to $\mathscr{L} \setminus \mathscr{S}$ are more complicated and were explored in [42, 88], [50, Section 3.7]. We direct the reader to [45, 31] and the references therein for more specific examples.

For a more detailed review of the heavy-tailed distributions and their properties, refer to the monographs of Bingham et al. [16], Embrechts et al. [44], Foss et al. [50] and Konstantinides [71].

The classes of heavy tailed distributions can be characterised by the specific indices. The first one is the so-called *upper Matuszewska index* introduced in [81] and later used in numerous studies investigating heavy-tail phenomena (e.g., [16, Section 2], [31, 106, 25, 53, 130, 132, 62] among others). For a d.f. F, the *upper Matuszewska index* J_F^+ is defined by

$$\begin{aligned} J_F^+ &:= \inf_{y>1} \left\{ -\frac{1}{\log y} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right\} \\ &= -\lim_{y\to\infty} \frac{1}{\log y} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}. \end{aligned} \quad (2.10)$$

The second important index is the so-called *L-index* used in e.g., [120, 131, 130, 132, 62, 76]. For a d.f. F, the *L-index* L_F is

$$L_F := \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}. \quad (2.11)$$

Definitions of the aforementioned classes of heavy-tailed distributions as well as (2.10) and (2.11) imply the following characterisations (see also [106]):

$$\begin{aligned} &F \in \mathscr{D} &&\iff && J_F^+ < \infty &&\iff && L_F > 0; \\ &F \in \mathscr{C} &&\iff && L_F = 1 &&\implies && J_F^+ < \infty; \end{aligned}$$

$$F \in \mathscr{R}_\alpha \qquad \Longrightarrow \qquad L_F = 1, \quad J_F^+ = \alpha.$$

In the dissertation we will consider one additional class of functions satisfying a certain tail monotonicity condition.

Definition 2.8. *A d.f. F supported on $\mathbb{R}$ is said to have a* positively decreasing *tail, written as $F \in \mathscr{P}_\mathscr{D}$, if for any fixed $y \in (0,1)$,*

$$\liminf_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} > 1,$$

or, equivalently,

$$\limsup_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} < 1$$

for any $y > 1$.

The class of positively decreasing functions was considered in [16, 75, 69] among others. For a more general treatment of functions, not limited to the distribution functions satisfying the condition described in Definition 2.8, refer to [20, 21, 19]. We observe that

$$\mathscr{R}_+ := \bigcup_{\alpha>0} \mathscr{R}_\alpha \subset \mathscr{P}_\mathscr{D}; \quad \mathscr{R}_0 \cap \mathscr{P}_\mathscr{D} = \emptyset; \quad \mathscr{C} \cap \mathscr{P}_\mathscr{D} \neq \emptyset; \quad \mathscr{P}_\mathscr{D} \setminus \mathscr{H} \neq \emptyset.$$

The first three relations follow immediately from Definitions 2.2, 2.4, and 2.8. To see that the class $\mathscr{P}_\mathscr{D}$ contains light-tailed distributions, it suffices to take a light-tailed exponential distribution (see Example 2.2). Indeed, for $\overline{F}(x) = \mathrm{e}^{-\lambda x}$, $\lambda > 0$, we have

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \limsup_{x\to\infty} \frac{\mathrm{e}^{-\lambda xy}}{\mathrm{e}^{-\lambda x}} = \limsup_{x\to\infty} \mathrm{e}^{-\lambda x(y-1)} < 1,$$

provided that $y > 1$. Hence, $F \in \mathscr{P}_\mathscr{D} \setminus \mathscr{H}$.

2.2 Dependence structures

Since we are dealing with not necessarily independent r.v.s we introduce the main dependence structure used in the dissertation and briefly review the related structures (see also the discussion in Subsection 4.1). We note that all the dependence conditions considered concern the tail behaviour of the distributions and impose some sort of asymptotic tail independence regardless of the behaviour in non-extreme regions.

We begin with a dependence structure first introduced by Geluk and Tang in [56] as *Assumption A*, which was later referred to as *strong quasi-asymptotic independence* in other articles (e.g., [77, 116, 62, 76]).

Definition 2.9 (pSQAI structure). *Real-valued r.v.s $\xi_1, \dots, \xi_n$, $n \in \mathbb{N}$, with distributions supported on $\mathbb{R}$ are called pairwise* strongly quasi-asymptotically independent (pSQAI) *if for all pairs of indices $k, l \in \{1, \dots, n\}$, $k \neq l$,*

$$\lim_{\min\{x_k, x_l\} \to \infty} \mathbb{P}\left(|\xi_k| > x_k \mid \xi_l > x_l\right) = 0. \tag{2.12}$$

The following two dependence structures were later considered by Cheng in [26] (see also the references therein).

Definition 2.10 (pTQAI structure). *Real-valued r.v.s $\xi_1, \dots, \xi_n$, $n \in \mathbb{N}$, with distributions supported on $\mathbb{R}$ are called pairwise* tail quasi-asymptotically independent (pTQAI) *if for all pairs of indices $k, l \in \{1, \dots, n\}$, $k \neq l$,*

$$\begin{aligned} &\lim_{\min\{x_k, x_l\} \to \infty} \frac{\mathbb{P}(\xi_k^+ > x_k, \xi_l^+ > x_l)}{\overline{F}_{\xi_k^+}(x_k) + \overline{F}_{\xi_l^+}(x_l)} \\ &\qquad = \lim_{\min\{x_k, x_l\} \to \infty} \frac{\mathbb{P}(\xi_k^- > x_k, \xi_l^+ > x_l)}{\overline{F}_{\xi_k^+}(x_k) + \overline{F}_{\xi_l^+}(x_l)} = 0. \end{aligned} \tag{2.13}$$

Definition 2.11 (pAI structure). *Real-valued r.v.s $\xi_1, \dots, \xi_n$, $n \in \mathbb{N}$, with distributions supported on $\mathbb{R}$ are called pairwise* asymptotically independent (pAI) *if for all pairs of indices $k, l \in \{1, \dots, n\}$, $k \neq l$,*

$$\lim_{x \to \infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\overline{F}_{\xi_k}(x)} = \lim_{x \to \infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\overline{F}_{\xi_l}(x)} = 0. \tag{2.14}$$

Leipus et al. in [76] considered a new dependence structure similar to pSQAI, called *Assumption B*.

Definition 2.12 (Assumption B). *Real-valued r.v.s $\xi_1, \dots, \xi_n$, $n \in \mathbb{N}$, with distributions supported on $\mathbb{R}$ satisfy* Assumption B *if for all pairs of indices $k, l \in \{1, \dots, n\}$, $k \neq l$,*

$$\lim_{x \to \infty} \sup_{u \geqslant x} \mathbb{P}\left(\xi_k^+ > x \mid \xi_l^+ > u\right) = \lim_{x \to \infty} \sup_{u \geqslant x} \mathbb{P}\left(\xi_k^- > x \mid \xi_l^+ > u\right)$$

$$= \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}\left(\xi_k^+ > x \mid \xi_l^- > u\right) = 0.$$

The main dependence assumption about r.v.s $\xi_1, \ldots, \xi_n$ considered in this dissertation, the so-called *pairwise quasi-asymptotic independence* due to Chen and Yuen [25], is now introduced.

Definition 2.13 (pQAI structure)**.** *Real-valued r.v.s $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, with distributions supported on $\mathbb{R}$ are called pairwise* quasi-asymptotically independent (pQAI) *if for all pairs of indices $k, l \in \{1, \ldots, n\}$, $k \neq l$,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^+ > x)}{\overline{F}_{\xi_k^+}(x) + \overline{F}_{\xi_l^+}(x)} = \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^- > x)}{\overline{F}_{\xi_k^+(x)} + \overline{F}_{\xi_l^+}(x)} = 0. \tag{2.15}$$

It turns out that the pQAI condition imposes the least restrictions out of the dependence structures considered in this section, as is summarised in the following proposition.

Proposition 2.1. *Let $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, be real-valued r.v.s with distributions supported on $\mathbb{R}$. Then*

$$\text{Assumption B} \implies \text{pSQAI} \implies \text{pTQAI} \implies \text{pQAI}. \tag{2.16}$$

Moreover, if the r.v.s $\xi_1, \ldots, \xi_n$ satisfy

$$\lim_{x\to\infty} \frac{\overline{F}_{\xi_k^-}(x)}{\overline{F}_{\xi_k}(x)} = 0 \qquad \text{for all} \qquad k \in \{1, \ldots, n\}, \tag{2.17}$$

then pAI $\implies$ pQAI*.*

Remark 2.1. *The additional condition* (2.17) *is not accidental—while the* pAI *structure can be considered on its own, it does not impose any restrictions on the asymptotic behaviour of the left tails of r.v.s $\xi_1, \ldots, \xi_n$, unlike the* pQAI *or* pSQAI *conditions. What is more, in [26] the condition* (2.17) *was considered together with the* pAI *structure (see also the discussion in Section 4.1).*

Proof. We first prove the (2.16).

(Case: Assumption B $\implies$ pSQAI) Assume that the r.v.s $\xi_1, \ldots, \xi_n$

satisfy Assumption B. For any pair ξ_k, ξ_l, $1 \leqslant k \neq l \leqslant n$, we have

$$\begin{aligned}
&\lim_{\min\{x_k,x_l\}\to\infty} \mathbb{P}\left(|\xi_k| > x_k \mid \xi_l > x_l\right)\\
&\quad = \lim_{\min\{x_k,x_l\}\to\infty} \left(\mathbb{P}\left(\xi_k > x_k \mid \xi_l > x_l\right) + \mathbb{P}\left(\xi_k^- > x_k \mid \xi_l > x_l\right)\right)\\
&\quad \leqslant \lim_{\min\{x_k,x_l\}\to\infty} \mathbb{P}\left(\xi_k > \min\{x_k,x_l\} \mid \xi_l > x_l\right)\\
&\qquad\qquad + \lim_{\min\{x_k,x_l\}\to\infty} \mathbb{P}\left(\xi_k^- > \min\{x_k,x_l\} \mid \xi_l > x_l\right)\\
&\quad \leqslant \lim_{\min\{x_k,x_l\}\to\infty} \sup_{u\geqslant\min\{x_k,x_l\}} \mathbb{P}\left(\xi_k > \min\{x_k,x_l\} \mid \xi_l > u\right)\\
&\qquad\qquad + \lim_{\min\{x_k,x_l\}\to\infty} \sup_{u\geqslant\min\{x_k,x_l\}} \mathbb{P}\left(\xi_k^- > \min\{x_k,x_l\} \mid \xi_l > u\right)\\
&\quad \leqslant \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}\left(\xi_k > x \mid \xi_l > u\right)\\
&\qquad\qquad + \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}\left(\xi_k^- > x \mid \xi_l > u\right) = 0.
\end{aligned}$$

(Case: pSQAI $\Longrightarrow$ pTQAI) Assume that the r.v.s $\xi_1, \ldots, \xi_n$ are pSQAI. For any pair ξ_k, ξ_l, $1 \leqslant k \neq l \leqslant n$, we have

$$\begin{aligned}
&\lim_{\min\{x_k,x_l\}\to\infty} \frac{\mathbb{P}(\xi_k^+ > x_k, \xi_l^+ > x_l)}{\overline{F}_{\xi_k^+}(x_k) + \overline{F}_{\xi_l^+}(x_l)}\\
&\qquad \leqslant \lim_{\min\{x_k,x_l\}\to\infty} \mathbb{P}\left(\xi_k > x_k \mid \xi_l > x_l\right) = 0.
\end{aligned}$$

In the same way,

$$\begin{aligned}
&\lim_{\min\{x_k,x_l\}\to\infty} \frac{\mathbb{P}(\xi_k^- > x_k, \xi_l^+ > x_l)}{\overline{F}_{\xi_k^+}(x_k) + \overline{F}_{\xi_l^+}(x_l)}\\
&\qquad \leqslant \lim_{\min\{x_k,x_l\}\to\infty} \mathbb{P}\left(\xi_k^- > x_k \mid \xi_l > x_l\right) = 0.
\end{aligned}$$

(Case: pTQAI $\Longrightarrow$ pQAI) The case follows immediately, by taking $x_k = x_l$ in (2.13).

To prove the implication pAI with (2.17) $\Longrightarrow$ pQAI, observe, that for any ξ_k, ξ_l, $1 \leqslant k \neq l \leqslant n$,

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\overline{F}_{\xi_k}(x) + \overline{F}_{\xi_l}(x)} \leqslant \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\overline{F}_{\xi_k}(x)} = 0$$

and

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, {\xi_l}^- > x)}{\overline{F}_{\xi_k}(x) + \overline{F}_{\xi_l}(x)} \leqslant \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, {\xi_l}^- > x)}{\overline{F}_{\xi_k}(x)} \leqslant \lim_{x\to\infty} \frac{\mathbb{P}({\xi_l}^- > x)}{\overline{F}_{\xi_k}(x)}$$
$$= \lim_{x\to\infty} \frac{F_{\xi_k}(-x)}{\overline{F}_{\xi_k}(x)} = 0.$$

□

Examples of dependent r.v.s could be constructed using copulas. Before proceeding with further details, we briefly recall the relevant theoretical background.

Definition 2.14. *(See [84, Definition 2.2.2]) A* bivariate *or* 2-dimensional copula *is a function* $C\colon [0{,}1]^2 \to [0{,}1]$*, with the following properties:*

- $C(x,0) = C(0,x) = 0 \qquad$ *for all* $\qquad x \in [0{,}1]$;
- $C(x,1) = C(1,x) = x \qquad$ *for all* $\qquad x \in [0{,}1]$;
- *for all* $0 \leqslant u_1 \leqslant u_2 \leqslant 1$ *and* $0 \leqslant v_1 \leqslant v_2 \leqslant 1$,

$$C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geqslant 0.$$

Definition 2.14 implies that bivariate copulas are basically 2-dimensional distributions with uniform marginal distributions. Below we list three copulas that will be used throughout the dissertation.

Example 2.6 (FGM copula)**.** *A bivariate* Farlie-Gumbel-Morgenstern (FGM) *copula [84, Example 3.12] with parameter* $\vartheta \in [-1,1]$ *is defined by*

$$C_\vartheta(u,v) = uv + \vartheta uv(1-u)(1-v), \quad u,v \in [0,1]. \tag{2.18}$$

The corresponding joint density function is

$$c_\vartheta(u,v) := \frac{\partial^2 C_\vartheta(u,v)}{\partial u \partial v} = 1 + \vartheta(1-2u)(1-2v).$$

The plots of $c_\vartheta(u,v)$ *for* $\vartheta \in \{0.5, -0.5\}$ *are shown in Figure 2.2.*

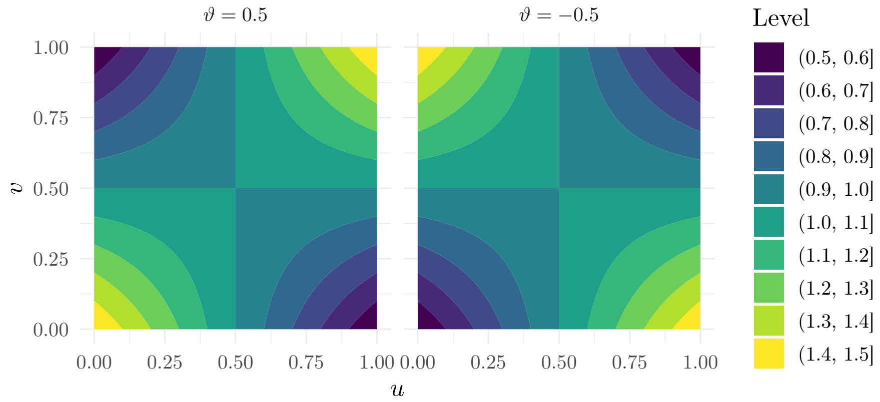

Figure 2.2: Density plot of FGM copula $C_\vartheta(u, v)$

Example 2.7 (AMH copula)**.** *A bivariate* Ali-Mikhail-Haq (AMH) *copula [4] with parameter* $\vartheta \in (-1, 1)$ *is defined by*

$$C_\vartheta(u, v) = \frac{uv}{1 - \vartheta(1-u)(1-v)}, \quad u, v \in [0, 1]. \tag{2.19}$$

The corresponding joint density function is

$$c_\vartheta(u, v) = \frac{\partial^2 C_\vartheta(u, v)}{\partial u \partial v} = \frac{(1-\vartheta)(1-\vartheta(1-u)(1-v)) + 2\vartheta uv}{(1-\vartheta(1-u)(1-v))^3}.$$

The plots of $c_\vartheta(u, v)$ *for* $\vartheta \in \{0.5, -0.5\}$ *are shown in Figure 2.3.*

Example 2.8 (Frank copula)**.** *A bivariate* Frank *copula [84, Section 4.2] with parameter* $\vartheta \in \mathbb{R} \setminus \{0\}$ *is defined by*

$$C_\vartheta(u, v) = -\frac{1}{\vartheta} \log\left(1 + \frac{(\mathrm{e}^{-\vartheta u} - 1)(\mathrm{e}^{-\vartheta v} - 1)}{\mathrm{e}^{-\vartheta} - 1}\right), \quad u, v \in [0, 1]. \tag{2.20}$$

The corresponding joint density function is

$$c_\vartheta(u, v) = \frac{\partial^2 C_\vartheta(u, v)}{\partial u \partial v} = \frac{\vartheta \mathrm{e}^{-\vartheta(u+v)}(1 - \mathrm{e}^{-\vartheta})}{\mathrm{e}^{-\vartheta(u+v)} - \mathrm{e}^{-\vartheta u} - \mathrm{e}^{-\vartheta v} + \mathrm{e}^{-\vartheta}}.$$

The plots of $c_\vartheta(u, v)$ *for* $\vartheta \in \{5, -5\}$ *are shown in Figure 2.4.*

The following theorem due to Sklar [100] is the crucial result that

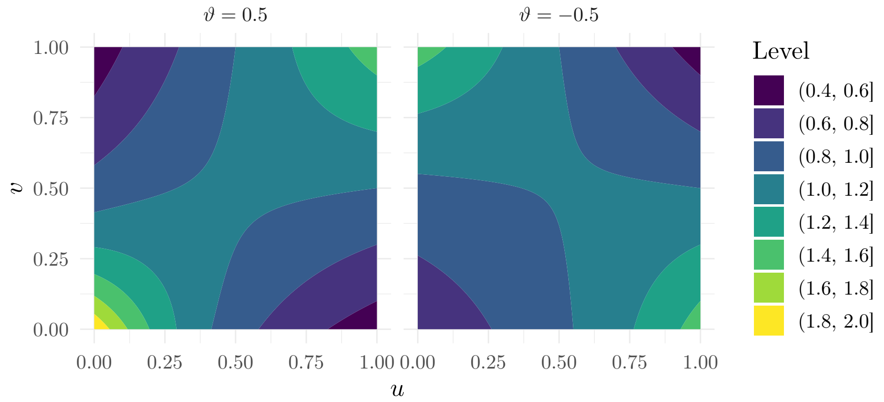

Figure 2.3: Density plot of AMH copula $C_\vartheta(u, v)$

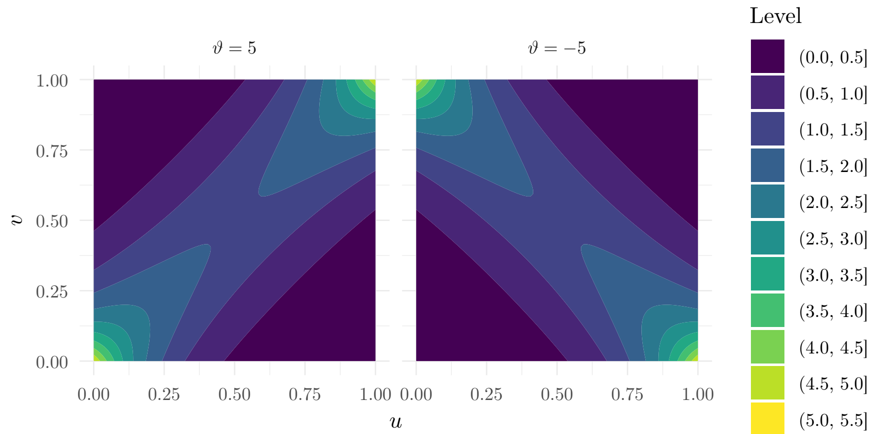

Figure 2.4: Density plot of Frank copula $C_\vartheta(u, v)$

links multivariate distributions to their marginals and copulas (see [84, Theorems 2.3.3 and 2.4.1]).

Theorem 2.1 (Sklar's Theorem)**.** *Let F be a bivariate distribution function with marginal d.f.s F_1 and F_2. Then there exists a bivariate copula C, such that*

$$F(x_1, x_2) = C\left(F_1(x_1), F_2(x_2)\right) \quad \textit{for all} \quad x_1, x_2 \in \overline{\mathbb{R}}. \tag{2.21}$$

If the d.f.s F_1 and F_2 are continuous, then the copula C is uniquely *determined. Otherwise, C is only uniquely determined on* $\mathrm{Ran}(F_1) \times \mathrm{Ran}(F_2)$. *Conversely, let C be a bivariate copula and let F_1, F_2 be two d.f.s. Then there exists a bivariate d.f F defined by* (2.21) *with marginal d.f.s F_1 and F_2.*

It can be shown that the FGM, AMH and Frank copulas imply the QAI dependence structure via Sklar's theorem.

Example 2.9. *Let ξ_1 and ξ_2 be r.v.s with corresponding d.f.s F_1 and F_2 supported on $\mathbb{R}$, such that $\overline{F}_1(x) > 0$ and $\overline{F}_2(x) > 0$ for all $x > 0$. Consider the* FGM *copula C_ϑ given in* (2.18). *It follows from Sklar's theorem (see Theorem 2.1) that the function $F(x_1, x_2) := C_\vartheta(F_1(x_1), F_2(x_2))$ is a bivariate d.f. with marginal d.f.s F_1, F_2. Then, for large enough x,*

$$\begin{aligned}
\frac{\mathbb{P}(\xi_1^+ > x, \xi_2^+ > x)}{\overline{F}_1(x) + \overline{F}_2(x)} &= \frac{1 - F_1(x) - F_2(x) + C_\vartheta(F_1(x), F_2(x))}{\overline{F}_1(x) + \overline{F}_2(x)} \\
&= \frac{\overline{F}_1(x)\overline{F}_2(x)\left(1 + \vartheta F_1(x) F_2(x)\right)}{\overline{F}_1(x) + \overline{F}_2(x)} \\
&\leqslant \frac{\overline{F}_1(x)\overline{F}_2(x)}{\overline{F}_1(x) + \overline{F}_2(x)} + \frac{\overline{F}_1(x)\overline{F}_2(x)F_1(x)F_2(x)}{\overline{F}_1(x) + \overline{F}_2(x)} \\
&\leqslant 2\overline{F}_1(x). \qquad (2.22)
\end{aligned}$$

Similarly, by observing that

$$\begin{aligned}
\mathbb{P}(\xi_1^+ > x, \xi_2^- > x) \leqslant \mathbb{P}(\xi_1^+ > x, \xi_2^- \geqslant x) &= \mathbb{P}(\xi_1^+ > x, \xi_j \leqslant -x) \\
&= \mathbb{P}(\xi_2^+ \leqslant -x) - \mathbb{P}(\xi_1^+ \leqslant x, \xi_2^+ \leqslant -x)
\end{aligned}$$

for positive x, we get for large enough x that

$$\begin{aligned}
\frac{\mathbb{P}(\xi_1^+ > x, \xi_2^- > x)}{\overline{F}_1(x) + \overline{F}_2(x)} &\leqslant \frac{F_2(-x) - C_\vartheta\left(F_1(x), F_2(-x)\right)}{\overline{F}_1(x) + \overline{F}_2(x)} \\
&= \frac{\overline{F}_1(x)F_2(-x)\left(1 - \vartheta F_1(x) F_2(-x)\right)}{\overline{F}_1(x) + \overline{F}_2(x)} \\
&= \frac{\overline{F}_1(x)F_2(-x)}{\overline{F}_1(x) + \overline{F}_2(x)} + \frac{\overline{F}_1(x)F_2(-x)F_1(x)F_2(-x)}{\overline{F}_1(x) + \overline{F}_2(x)} \\
&\leqslant 2F_2(-x). \qquad (2.23)
\end{aligned}$$

The estimates (2.22) *and* (2.23) *imply* (2.15). *Consequently, the r.v.s ξ_1 and ξ_2 are* QAI.

Example 2.10. *Let ξ_1 and ξ_2 be r.v.s with corresponding d.f.s F_1 and F_2 supported on $\mathbb{R}$, such that $\overline{F}_1(x) > 0$ and $\overline{F}_2(x) > 0$ for all $x > 0$. Consider the* AMH *copula C_ϑ given in* (2.19)*. It follows from Sklar's theorem (see Theorem 2.1) that the function $F(x_1, x_2) := C_\vartheta(F_1(x_1), F_2(x_2))$ is a bivariate d.f. with marginal d.f.s F_1, F_2. Then, for large enough x,*

$$\begin{aligned}\frac{\mathbb{P}(\xi_1^+ > x, \xi_2^+ > x)}{\overline{F}_1(x) + \overline{F}_2(x)} &= \frac{1 - F_1(x) - F_2(x) + C_\vartheta(F_1(x), F_2(x))}{\overline{F}_1(x) + \overline{F}_2(x)} \\ &= \frac{\overline{F}_1(x)\overline{F}_2(x)\,(1 + \vartheta F_1(x)F_2(x))}{\overline{F}_1(x) + \overline{F}_2(x)} \\ &= \frac{\overline{F}_1(x)\overline{F}_2(x)(1 + \vartheta(F_2(x) - \overline{F}_1(x)))}{(\overline{F}_1(x) + \overline{F}_2(x))(1 - \vartheta\overline{F}_1(x)\overline{F}_2(x))} \\ &\leqslant \frac{2\overline{F}_2(x)}{1 - \vartheta\overline{F}_1(x)\overline{F}_2(x)}. \end{aligned} \tag{2.24}$$

In the same fashion, for sufficiently large x, we obtain

$$\begin{aligned}\frac{\mathbb{P}(\xi_1^+ > x, \xi_2^- > x)}{\overline{F}_1(x) + \overline{F}_2(x)} &\leqslant \frac{F_2(-x) - C_\vartheta\left(F_1(x), F_2(-x)\right)}{\overline{F}_1(x) + \overline{F}_2(x)} \\ &= \frac{F_2(-x)\overline{F}_1(x)(1 - \vartheta\overline{F}_2(-x))}{(\overline{F}_1(x) + \overline{F}_2(x))(1 - \vartheta\overline{F}_1(x)\overline{F}_2(-x))} \\ &\leqslant \frac{2F_2(-x)}{1 - \vartheta\overline{F}_1(x)\overline{F}_2(-x)}. \end{aligned} \tag{2.25}$$

The derived estimates (2.24) *and* (2.25) *imply that the r.v.s ξ_1 and ξ_2 are* QAI.

Example 2.11. *Let ξ_1 and ξ_2 be r.v.s with corresponding d.f.s F_1 and F_2 supported on $\mathbb{R}$, such that $\overline{F}_1(x) > 0$ and $\overline{F}_2(x) > 0$ for all $x > 0$. Consider the Frank copula C_ϑ given in* (2.20)*. It follows from Sklar's theorem (see Theorem 2.1) that the function $F(x_1, x_2) := C_\vartheta(F_1(x_1), F_2(x_2))$ is a bivariate d.f. with marginal d.f.s F_1, F_2. As stated in [84, Section 4.2] the Frank copula satisfies the property of radial symmetry:*

$$C_\vartheta(u, v) = u + v - 1 + C_\vartheta(1 - u, 1 - v), \quad u, v \in [0, 1],$$

thus implying

$$\begin{aligned}\mathbb{P}(\xi_1^+ > x, \xi_2^+ > x) &= 1 - F_1(x) - F_2(x) + C_\vartheta(F_1(x), F_2(x)) \\ &= \overline{F}_1(x) + \overline{F}_2(x) - 1 + C_\vartheta(1 - \overline{F}_1(x), 1 - \overline{F}_2(x))\end{aligned}$$

$$
\begin{aligned}
&= C_\vartheta(\overline{F}_1(x), \overline{F}_2(x)) \\
&= -\frac{1}{\vartheta}\log\left(1 + \frac{(\mathrm{e}^{-\vartheta\overline{F}_1(x)} - 1)(\mathrm{e}^{-\vartheta\overline{F}_2(x)} - 1)}{\mathrm{e}^{-\vartheta} - 1}\right)
\end{aligned}
\tag{2.26}
$$

for large enough x. It follows from the relations $\mathrm{e}^{-\vartheta\overline{F}_i(x)} - 1 = -\vartheta\overline{F}_i(x) + o(\overline{F}_i(x))$, $i = \{1,2\}$, and (2.26) that

$$
\begin{aligned}
\mathbb{P}(\xi_1^+ > x, \xi_2^+ > x) &= -\frac{1}{\vartheta}\log\left(1 + \frac{\vartheta^2\overline{F}_1(x)\overline{F}_2(x)}{\mathrm{e}^{-\vartheta} - 1} + o(\overline{F}_1(x)\overline{F}_2(x))\right) \\
&= -\frac{\vartheta}{\mathrm{e}^{-\vartheta - 1}}\overline{F}_1(x)\overline{F}_2(x) + o(\overline{F}_1(x)\overline{F}_2(x)).
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
\frac{\mathbb{P}(\xi_1^+ > x, \xi_2^+ > x)}{\overline{F}_1(x) + \overline{F}_2(x)} &= -\frac{\vartheta}{\mathrm{e}^{-\vartheta - 1}}\frac{\overline{F}_1(x)\overline{F}_2(x)}{\overline{F}_1(x) + \overline{F}_2(x)} \\
&\quad + o\left(\frac{\overline{F}_1(x)\overline{F}_2(x)}{\overline{F}_1(x) + \overline{F}_2(x)}\right).
\end{aligned}
\tag{2.27}
$$

Similarly, for sufficiently large x, we have

$$
\begin{aligned}
\mathbb{P}(\xi_1^+ > x, \xi_2^- > x) &= \mathbb{P}(\xi_1^+ > x, \xi_2 < -x) \leqslant \mathbb{P}(\xi_1 > x, \xi_2 \leqslant -x) \\
&= \overline{F}_1(x) - \mathbb{P}(\xi_1 > x, \xi_2 > -x) \\
&= \overline{F}_1(x) - C_\vartheta(\overline{F}_1(x), \overline{F}_2(-x)) \\
&= \overline{F}_1(x) + \frac{1}{\vartheta}\log\left(1 + \frac{(\mathrm{e}^{-\vartheta\overline{F}_1(x)} - 1)(\mathrm{e}^{-\vartheta\overline{F}_2(-x)} - 1)}{\mathrm{e}^{-\vartheta} - 1}\right) \\
&= \overline{F}_1(x) + \frac{1}{\vartheta}\log\left(1 - \frac{\vartheta\overline{F}_1(x)(\mathrm{e}^{-\vartheta\overline{F}_2(-x)} - 1)}{\mathrm{e}^{-\vartheta} - 1} + o(\overline{F}_1(x))\right) \\
&= \overline{F}_1(x) - \frac{\overline{F}_1(x)(\mathrm{e}^{-\vartheta\overline{F}_2(-x)} - 1)}{\mathrm{e}^{-\vartheta} - 1} + o(\overline{F}_1(x)).
\end{aligned}
$$

Again, for large enough x, we obtain

$$
\begin{aligned}
\frac{\mathbb{P}(\xi_1^+ > x, \xi_2^- > x)}{\overline{F}_1(x) + \overline{F}_2(x)} &\leqslant \frac{\overline{F}_1(x) - \frac{\overline{F}_1(x)(\mathrm{e}^{-\vartheta\overline{F}_2(-x)} - 1)}{\mathrm{e}^{-\vartheta} - 1} + o(\overline{F}_1(x))}{\overline{F}_1(x) + \overline{F}_2(x)} \\
&\leqslant 1 - \frac{e^{\vartheta\overline{F}_2(-x) - 1}}{\mathrm{e}^{-\vartheta} - 1} + o(1).
\end{aligned}
\tag{2.28}
$$

From the estimates (2.27) *and* (2.28) *it follows that the r.v.s* ξ_1 *and* ξ_2 *are* QAI.

In a setting where a collection of three or more r.v.s is considered, there are multiple ways to impose the pQAI dependence structure. Below we consider a couple of examples.

Example 2.12 (Mixing pairwise dependence with pairwise independence)**.** *Let* $\xi_1, \ldots, \xi_n$ *be r.v.s with corresponding d.f.s* $F_1, \ldots, F_n$ *supported on* $\mathbb{R}$*, such that* $\overline{F}_i(x) > 0$*,* $i = 1, \ldots, n$*, for all* $x > 0$*. Additionally, let any two r.v.s* ξ_i, ξ_j*,* $i, j = 1, \ldots, n$*,* $i \neq j$ *have one of the joint distribution functions considered in Examples 2.9, 2.10, 2.11 if* $\max\{i,j\} - \min\{i,j\} = 1$*,* $\min\{i,j\} = 2k-1$*,* $k \in \mathbb{N}$*, and be independent otherwise. We have already shown that in the first case each pair consists of* QAI *r.v.s. In the other case, the r.v.s are* QAI *as well, since independence implies the* QAI *structure.*

Example 2.13 (Multivariate FGM distribution)**.** *Let* $\xi_1, \ldots, \xi_n$ *be r.v.s with corresponding d.f.s* $F_1, \ldots, F_n$ *supported on* $\mathbb{R}$*, such that* $\overline{F}_i(x) > 0$*,* $i = 1, \ldots, n$*, for all* $x > 0$*. A multivariate* FGM *distribution (see [72, Chapter 44, Section 13]) is defined by*

$$\begin{aligned}\mathbb{P}\left(\xi_1 \leqslant x_1, \ldots, \xi_n \leqslant x_n\right) = {} & \prod_{i=1}^{n} F_i(x_i) \\ & \times \left(1 + \sum_{1 \leqslant i < j \leqslant n} \vartheta_{ij} \overline{F}_i(x_i) \overline{F}_j(x_j)\right),\end{aligned} \tag{2.29}$$

where parameters ϑ_{ij}*,* $i, j \in \{1, \ldots, n\}$*, satisfy an addition condition*

$$1 + \sum_{1 \leqslant i < j \leqslant n} \varepsilon_i \varepsilon_j \vartheta_{ij} \geqslant 0 \tag{2.30}$$

for all $\varepsilon_i = -\sup\{x \in \mathbb{R} \colon F_i(x)\} \setminus \{0,1\}$ *or* $\varepsilon_i = 1 - \inf\{x \in \mathbb{R} \colon F_i(x)\} \setminus \{0,1\}$*. The condition* (2.30) *ensures that* (2.29) *is a well-defined d.f. The multivariate* FGM *distribution can be constructed using the multivariate* FGM *copula (see [84, Theorem 2.10.9.]) and applying the multivariate version of Sklar's theorem (see [84, Example 3.31.]), similarly as in the bivariate case considered in Example 2.9. Any 2-dimensional marginal distribution of a multivariate* FGM *copula is a bivariate* FGM *copula [84,* *Example 3.31.]. Hence, for any given pair $\{\xi_i, \xi_j\}$, $i \neq j$, $i, j = 1, \ldots, n$, Example 2.9 applies implying the* QAI *structure.*

For more information on the application of copulas to problems involving the modelling of dependence between heavy-tailed distributions, the reader may refer to Albrechter et al. [3], Asimit et al. [10], Yang et al. [127], Wang et al. [118] and the references therein. For a systematic treatment of copulas theory, see, for instance, the books of Nelsen [84] and Durante and Sempi [41].

2.3 Quantile functions

In this subsection, we briefly recall some definitions and properties of generalised inverses of distribution functions. These will be employed in Sections 4.4, 5.4, 3.4, when performing MC simulations, and used extensively in Chapter 5.

Definition 2.15. *For a real-valued r.v. X with d.f. F, the* generalised inverse*, or* quantile function*, is a function $F^{\leftarrow}\colon [0,1] \to \overline{\mathbb{R}}$ defined by*

$$F^{\leftarrow}(q) := \inf\{x \in \mathbb{R} : F(x) \geqslant q\} = \inf\left\{x \in \mathbb{R} : \overline{F}(x) \leqslant 1-q\right\}, \quad (2.31)$$

where $F^{\leftarrow}(q) = \infty$ if $\{x \in \mathbb{R} : F(x) \geqslant q\} = \emptyset$.

Remark 2.2. *For a t.f. $\overline{F}$, the generalised inverse, or quantile function, is a function $\overline{F}^{\leftarrow} : [0,1] \to \overline{\mathbb{R}}$ defined by*

$$\overline{F}^{\leftarrow}(p) = \inf\left\{x \in \mathbb{R} : \overline{F}(x) \leqslant p\right\}, \quad (2.32)$$

where $\overline{F}^{\leftarrow}(p) = \infty$ if $\{x \in \mathbb{R} : \overline{F}(x) \leqslant p\} = \emptyset$. The definition follows directly from Definition 2.15 by observing that

$$\begin{aligned}\overline{F}^{\leftarrow}(p) &= \inf\left\{x \in \mathbb{R} : \overline{F}(x) \leqslant p\right\} \\ &= \inf\{x \in \mathbb{R} : F(x) \geqslant 1-p\} = F^{\leftarrow}(1-p).\end{aligned} \quad (2.33)$$

Below we list several useful properties of quantile functions.

Lemma 2.2. *Let X be a real-valued r.v. with d.f. F and corresponding quantile function $F^{\leftarrow}$. Then*

(1) *$F^{\leftarrow}(q) = -\infty$ if and only if $F(x) \geqslant q$ for all $x \in \mathbb{R}$. $\overline{F}^{\leftarrow}(p) = -\infty$ if and only if $\overline{F}(x) \leqslant p$ for all $x \in \mathbb{R}$.*

(2) $F^{\leftarrow}(q) = \infty$ *if and only if* $F(x) < q$ *for all* $x \in \mathbb{R}$. $\overline{F}^{\leftarrow}(p) = \infty$ *if and only if* $\overline{F}(x) > p$ *for all* $x \in \mathbb{R}$.

(3) $F^{\leftarrow}(F(x)) \leqslant x$ *and* $\overline{F}^{\leftarrow}(\overline{F}(x)) \leqslant x$. *If* F *is strictly increasing, then* $F^{\leftarrow}(F(x)) = x$ *and* $\overline{F}^{\leftarrow}(\overline{F}(x)) = x$.

(4) $F^{\leftarrow}(q) < \infty$ *implies* $F(F^{\leftarrow}(q)) \geqslant q$. $\overline{F}^{\leftarrow}(p) < \infty$ *implies* $\overline{F}(\overline{F}^{\leftarrow}(p)) \leqslant p$.

Proof. The proof of Lemma 2.2 for a d.f. F can be found in [43]. The cases (1)-(4) for a t.f. $\overline{F}$ follow almost directly by (2.33).

(1) $F^{\leftarrow}(p) = -\infty$ if and only if $F(x) \geqslant p$ for all $x \in \mathbb{R}$. Thus, $\overline{F}^{\leftarrow}(p) = F^{\leftarrow}(1-p) = -\infty$ if and only if $F(x) \geqslant 1-p$ or, equivalently, $\overline{F}(x) \leqslant p$ for all $x \in \mathbb{R}$.

(2) $F^{\leftarrow}(p) = \infty$ if and only if $F(x) < p$ for all $x \in \mathbb{R}$. Hence, $\overline{F}^{\leftarrow}(p) = F^{\leftarrow}(1-p) = -\infty$ if and only if $F(x) < 1-p$ or, equivalently, $\overline{F}(x) > p$ for all $x \in \mathbb{R}$.

(3) Relation (2.33) implies $F^{\leftarrow}(F(x)) = F^{\leftarrow}(1-\overline{F}(x)) = \overline{F}^{\leftarrow}(\overline{F}(x))$. Hence, the case follows immediately.

(4) $\overline{F}^{\leftarrow}(p) < \infty$ is equivalent to $F^{\leftarrow}(1-p) < \infty$ by (2.33). Thus, $F(F^{\leftarrow}(1-p)) \geqslant 1-p$ and the left hand side of the inequality can be expressed as

$$F(F^{\leftarrow}(1-p)) = F(\overline{F}^{\leftarrow}(p)) = 1 - \overline{F}(\overline{F}^{\leftarrow}(p))$$

completing the proof.

$\square$

One practical application of the quantile function is in MC simulation methods. The following lemma provides the theoretical justification.

Lemma 2.3 (Inverse probability integral transform)**.** *Let* $U \overset{d}{=} \mathcal{U}([0,1])$ *and* X *be a real-valued r.v. with d.f.* F. *Then* $F^{\leftarrow}(U) \overset{d}{=} X$.

One can find the proof of this lemma in [43]. By Lemma 2.3, one can easily generate samples from a given distribution F, provided that the expression of the generalised inverse $F^{\leftarrow}$ is known, since the majority of statistical programming languages allow users to sample from a uniform distribution.

In the following examples we derive the expressions of generalised inverses of distributions given in Examples 2.3, 2.4, 2.5.

Example 2.14 (Quantile function of Pareto d.f.)**.** *Consider a regularly varying Pareto d.f. F with parameters a, b, as given in* (2.1)*. Since F is strictly increasing on the interval $[0, \infty)$ for all $q \in (0,1)$, we have $F^{\leftarrow}(q) = F^{-1}(q)$ and*

$$q = F(x) = 1 - \left(\frac{b}{b+x}\right)^{a} \quad \Longleftrightarrow \quad F^{-1}(q) = b\,(1-q)^{-\frac{1}{a}} - b.$$

Thus,

$$F^{\leftarrow}(q) = b\,(1-q)^{-\frac{1}{a}} - b. \tag{2.34}$$

Example 2.15 (Quantile function of Peter and Paul d.f.)**.** *Consider a dominatedly varying Peter and Paul d.f. F with parameters a, b, as given in* (2.4)*. Since*

$$F(x) = \left(1 - \left(b^{-a}\right)^{\lfloor \log_b x \rfloor}\right) \mathbb{I}_{\{x \geqslant 1\}},$$

to obtain the generalised inverse $F^{\leftarrow}(q)$ we need to find the smallest x, for which $F(x) \geqslant q$. As

$$1 - (b^{-a})^{\lfloor \log_b x \rfloor} \geqslant q \quad \Longleftrightarrow \quad \lfloor \log_b x \rfloor \geqslant -\frac{1}{a}\log_b(1-q),$$

it follows that

$$F^{\leftarrow}(q) = b^{\left\lceil -\frac{1}{a}\log_b(1-q)\right\rceil}. \tag{2.35}$$

Example 2.16 (Quantile function of Cai-Tang d.f.)**.** *Consider a consistently varying Cai-Tang d.f. F with parameter p, as given in* (2.3)*. To obtain the generalised inverse $F^{\leftarrow}$, we observe that the d.f. F is continuous, strictly monotone on the interval $[1, \infty)$ and linearly increasing on the intervals $[2^k, 2^{k+1})$, $k \in \mathbb{N}_0$. Hence $F^{\leftarrow}$ coincides with F^{-1}.*

Suppose that for a given $q \in (0,1)$, the argument $x \in [2^k, 2^{k+1})$ is such that $F(x) = q$. In such a case, we have

$$F(x) = (1-p)\left(\frac{1-p^k}{1-p} + \left(\frac{x}{2^k} - 1\right)p^k\right) = q \quad \Longleftrightarrow$$

$$x = \frac{2^k \left(q - 1 + p^k(2-p)\right)}{p^k(1-p)}.$$

Since $F(2^n) = 1 - p^n$ *for all* $n \in \mathbb{N}_0$*, we obtain*

$$F^{\leftarrow}(q) = \frac{2^{\lfloor \log_p(1-q) \rfloor} \left(q - 1 + p^{\lfloor \log_p(1-q) \rfloor}(2-p)\right)}{p^{\lfloor \log_p(1-q) \rfloor}(1-p)} \tag{2.36}$$

for all $q \in [0, 1)$*.*

Similar to the case of univariate distributions, one can draw samples from a multivariate distribution even if its marginals are not necessarily mutually independent. The procedure is based on the so-called *Rosenblatt's* transformation presented in [95]. For a random vector $(X_1, \ldots, X_n)$ with an absolutely continuous distribution, the mapping is defined to the random vector $(U_1, \ldots, U_n)$, where $U_i \overset{d}{=} \mathcal{U}([0, 1])$, $i = 1, \ldots, n$, are i.i.d. r.v.s (for a generalised transformation without the absolute continuity assumption see [17]). Firstly, the n-variate distribution $F_{X_1,\ldots,X_n}$ of a vector $(X_1, \ldots, X_n)$ is expressed into the product of conditional distributions as follows:

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = F_1(x_1) F_2(x_2 \mid x_1) \cdots F_n(x_n \mid x_{n-1}, \ldots, x_1),$$

where $F_1(x_1) = \mathbb{P}(X_1 \leqslant x_1)$ and F_k, $k = 2, \ldots, n$, is the conditional distribution of X_k:

$$F_k(x_k \mid x_{k-1}, \ldots, x_1) = \mathbb{P}(X_k \leqslant x_k \mid X_{k-1} = x_{k-1}, \ldots, X_1 = x_1).$$

Then the r.v.s $F_1(X_1), F_2(X_2 \mid X_1), \ldots, F_n(X_n \mid X_{n-1}, \ldots, X_1)$ are independent and uniformly distributed on the interval $[0, 1]$.

To sample from the multivariate distribution of $(X_1, \ldots, X_n)$, $n > 1$, we follow the so-called *conditional distribution method* (see [36, Chapter 11, Section 1.2]) combined with Lemma 2.3.

Algorithm 1 (Multivariate distribution sampling). *Generation of samples from a multivariate distribution of a random vector* $(X_1, \ldots, X_n)$*.*

Step 1: *Generate* n *independent realisations* $u_1, \ldots, u_n$ *from the uniform distribution* $\mathcal{U}$([0,1])*.*

Step 2: *Obtain a realisation* $(x_1, \ldots, x_n)$ *from* $(X_1, \ldots, X_n)$ *recursively:*

$$\begin{aligned}
x_1 &= F_1^{\leftarrow}(u_1), \\
x_2 &= F_2^{\leftarrow}(u_2 \mid x_1), \\
&\vdots \\
x_n &= F_n^{\leftarrow}(u_n \mid x_{n-1}, \ldots, x_1).
\end{aligned}$$

For any copula $C(u_1, u_2) = \mathbb{P}(U_1 \leqslant u_1, U_2 \leqslant u_2)$, the conditional distribution function of U_1 given the event $\{U_2 = u_2\}$ is defined by

$$\begin{aligned}
C_{u_2}(u_1) :=& \, \mathbb{P}(U_1 \leqslant u_1 \mid U_2 = u_2) = \frac{\partial}{\partial u_2} C(u_1, u_2) \\
=& \lim_{\delta \to 0} \frac{C(u_1, u_2 + \delta) - C(u_1, u_2)}{\delta}.
\end{aligned} \tag{2.37}$$

By Theorem 2.2.7 of [84], it follows that the partial derivative in (2.37) exists almost everywhere on the interval $[0,1]$. Samples from a bivariate distribution defined by a bivariate copula and marginal distributions can be generated combining Sklar's theorem (see Theorem 2.1), Lemma 2.3 and the conditional distribution method discussed above. The procedure is summarised in the following algorithm (see also [84, Section 2.9]).

Algorithm 2. *Generation of samples from a bivariate distribution* F *characterised by marginal d.f.s* F_1, F_2 *and a copula* C.

Step 1: *Generate two independent realisations* t, u_2 *from* $\mathcal{U}([0,1])$.

Step 2: *To induce the copula implied dependence, transform* t *into* $u_1 = C_{u_2}^{\leftarrow}(t)$. *In this way, the realisation* (u_1, u_2) *from the copula* C *is obtained.*

Step 3: *Using Lemma 2.3, obtain the realisation from the bivariate distribution* F *by transforming* (u_1, u_2) *into* $(F_1^{\leftarrow}(u_1), F_2^{\leftarrow}(u_2))$.

In the following examples we derive the conditional distribution functions of the FGM, AMH and Frank copulas, given in (2.18), (2.19) and (2.20) respectively, and their generalised inverses.

Example 2.17. *Since the bivariate* FGM *copula in* (2.18) *is absolutely* *continuous, we get*

$$\begin{aligned} C_{\vartheta,u_2}(u_1) &:= \frac{\partial}{\partial u_2} C_\vartheta(u_1,u_2) \\ &= u_1 + \vartheta u_2(1-u_1)(1-u_2) - \vartheta u_1 u_2 (1-u_1) \\ &= u_1(1-\vartheta+2u_2\vartheta) + u_1^2(\vartheta - 2u_2\vartheta). \end{aligned}$$

It follows that the generalised inverse is

$$C^{\leftarrow}_{\vartheta,u_2}(q) = \frac{-(1-\vartheta+2u_2\vartheta) + \sqrt{(1-\vartheta+2u_2\vartheta)^2 - 4y(\vartheta-2u_2\vartheta)}}{2(\vartheta-2u_2\vartheta)}.$$

Example 2.18. *The bivariate* AMH *copula, given in* (2.19)*, is absolutely continuous and thus*

$$C_{\vartheta,u_2}(u_1) = \frac{u_1 - \vartheta u_1(1-u_1)(1-u_2) - \vartheta u_1 u_2 (1-u_1)}{(1-\vartheta(1-u_1)(1-u_2))^2}.$$

Then the generalised inverse is

$$\begin{aligned} C^{\leftarrow}_{\vartheta,u_2}(q) = {} & \frac{\vartheta q\left(\vartheta u_2^2 - 2\vartheta u_2 + \vartheta + u_2 - 1\right)}{\vartheta(\vartheta u_2^2 q - 2\vartheta u_2 q + \vartheta q - 1)} \\ & + \frac{1-\vartheta+\sqrt{4\vartheta^2 u_2^2 q - 4\vartheta^2 u_2 q + 4\vartheta u_2 q + \vartheta^2 - 2\vartheta + 1}}{2\vartheta(\vartheta u_2^2 q - 2\vartheta u_2 q + \vartheta q - 1)}. \end{aligned}$$

Example 2.19. *The bivariate Frank copula, given in* (2.20)*, is absolutely continuous and thus*

$$C_{\vartheta,u_2}(u_1) = \frac{\mathrm{e}^{-\vartheta u_2}(\mathrm{e}^{-\vartheta u_1} - 1)}{\mathrm{e}^{-\vartheta(u_1+u_2)} - \mathrm{e}^{-\vartheta u_1} - \mathrm{e}^{-\vartheta u_2} + \mathrm{e}^{-\vartheta}}.$$

It follows that the generalised inverse is

$$C^{\leftarrow}_{\vartheta,u_2}(q) = -\frac{1}{\vartheta}\log\left(\frac{\mathrm{e}^{-\vartheta u_2}(1-q) + q\,\mathrm{e}^{-\vartheta}}{\mathrm{e}^{-\vartheta u_2}(1-q)+q}\right).$$

2.4 Risk measures

In this section, we briefly recall the main facts about risk measures, which from a strictly mathematical perspective are real-valued functionals defined on a class of random variables representing random risks.

Definition 2.16. *A risk measure is a mapping $\rho\colon \mathcal{G} \to \mathbb{R}$, where $\mathcal{G}$ is a set of financial positions (risks).*

In practice, the interpretation of the risk measure ρ and the set $\mathcal{G}$ depends on the field they are used in. For instance, in finance and banking, risk measures are often employed to assess the risk of the portfolio of financial assets, like stocks or bonds, under market uncertainty. In such a case, a financial position $X \in \mathcal{G}$ can be the discounted value of the portfolio or the sum of its generated profit and economic capital. The associated risk $\rho(X)$ can then be interpreted as the minimum amount of cash (capital) that needs to be added and invested in risk-free assets to make the position acceptable[1] [9]. In an insurance context, r.v.s in $\mathcal{G}$ are often interpreted on a loss-profit scale, which means that a r.v. $X \in \mathcal{G}$ represents a potential loss insurer is facing, while $\rho(X)$ can denote the insurance premium charged [58].

It should be noted, though, that without loss of generality, financial positions can be interpreted either on a profit-loss or loss-profit scale. In particular, if $\rho\colon \mathcal{G} \to \mathbb{R}$ is a risk measure defined on a set $\mathcal{G}$ of profit-loss risks, then the functional $\pi(X) = \rho(-X)$, for all $X \in \mathcal{G}$ is a risk measure defined on a set of loss-profit risks. Hereafter, unless stated otherwise, we consider financial positions on a loss-profit scale.

In their seminal paper [9] Artzner and co-authors presented a systematic treatment of risk measures and provided a set of intuitively interpretable axioms or desirable properties that "should hold for any risk measure which is to be used to effectively regulate or manage risks"[2] (see Axioms TI, PH, S, M below).

Axiom TI (Translation invariance)**.** *For all $X \in \mathcal{G}$ and $\alpha \in \mathbb{R}$,*

$$\rho(X + \alpha) = \rho(X) + \alpha.$$

The translation invariance axiom means that if the potential loss was increased (decreased) by a fixed amount α almost surely, then the risk of the position would increase (decrease) by α.

[1]For simplicity, in this dissertation we consider random risks and risk measures without the impact of interest rates. In a more general setting, the results could be extended by adding discount factors.

[2]In [9] the axioms were formulated for random risks on a profit-loss scale.

Axiom PH (Positive homogeneity)**.** *For all $X \in \mathcal{G}$ and $\lambda \geqslant 0$,*

$$\rho(\lambda X) = \lambda \rho(X).$$

The positive homogeneity axiom states that rescaling a financial position by a positive factor rescales the risk of the position by the same factor. Moreover, it implies a natural property $\rho(0) = 0$, meaning that holding no financial positions results in zero risk.

Axiom S (Subadditivity)**.** *For all $X_1, X_2 \in \mathcal{G}$,*

$$\rho(X_1 + X_2) \leqslant \rho(X_1) + \rho(X_2).$$

As mentioned in [9], the intuition behind the subadditivity axiom is that an aggregation of risks does not result in additional risk. What is more, Axioms PH and S imply a more general convexity property, which, in the context of risk measures, was introduced by Föllmer and Schied in [48].

Axiom C (Convexity)**.** *For all $X_1, X_2 \in \mathcal{G}$ and $\lambda \in [0, 1]$,*

$$\rho\left(\lambda X_1 + (1-\lambda)X_2\right) \leqslant \lambda \rho(X_1) + (1-\lambda)\rho(X_2).$$

The convexity axiom directly reflects the principle of diversification in a financial portfolio; the risk of a diversified position does not exceed the weighted average of the risks of individual financial positions. To see that Axioms S and PH imply Axiom C, observe that

$$\begin{aligned} \rho\left(\lambda X_1 + (1-\lambda)X_2\right) &\underset{\text{(S)}}{\leqslant} \rho\left(\lambda X_1\right) + \rho\left((1-\lambda)X_2\right) \\ &\underset{\text{(PH)}}{=} \lambda \rho(X_1) + (1-\lambda)\rho(X_2). \end{aligned}$$

The following axiom from [9] encapsulates the concept that, when comparing two financial positions, the one that generates the larger losses is more risky.

Axiom M (Monotonicity)**.** *For all $X, Y \in \mathcal{G}$ satisfying $X \leqslant Y$,*

$$\rho(X) \leqslant \rho(Y).$$

Artzner et al. in [9] introduced the concept of a coherent risk measure for risk measures that satisfy the aforementioned axioms.

Definition 2.17 (Coherent risk measure)**.** *A risk measure $\rho\colon \mathcal{G} \to \mathbb{R}$ is called* coherent *if it satisfies Axioms TI, S, PH and M.*

The notion of coherence was later revisited and generalised by other authors as well; see [35, 48, 52, 103] and the references therein. Notably, Föllmer and Schied in [48] and Fritelli and Rosazza Gianin [52] in [52] introduced the so-called convex risk measures, an extension of coherence, by replacing Axioms PH and S with Axiom C (see also [52]). In [48] it was argued that the increase in risk resulting from a multiplicative position increase may be nonlinear, as stated in Axiom PH.

Definition 2.18. *Convex risk measure A risk measure $\rho\colon \mathcal{G} \to \mathbb{R}$ is called* convex *if it satisfies Axioms C, TI and M*

A more comprehensive overview of the properties and other classifications of risk measures is beyond the scope of this dissertation. The reader is referred to [11] for distortion risk measures, to [1, 2] for spectral risk measures, to [13] for expectile risk measures, and to [12, 135] for elicitable risk measures.

Some examples of risk measures

One of the most extensively used risk measures is Value-at-Risk (VaR), which is defined as follows.

Definition 2.19 (Value-at-Risk)**.** *Let $q \in (0, 1)$ and $X \in \mathcal{G}$. The* Value-at-Risk (VaR) *risk measure at level q for the variable X is*

$$\mathrm{VaR}_q(X) := \inf\{x \in \mathbb{R} : \mathbb{P}(X > x) \leqslant 1 - q\} = F_X^{\leftarrow}(q).$$

It can be seen from the definition, that VaR_q is basically the q-th percentile of a loss distribution. Due to its simplicity and ease of computation the measure has quickly became an industry standard. Notwithstanding the simplicity of VaR, certain shortcomings were identified (see [9], [126]). First, it was shown in [9] that Value-at-Risk is not a coherent risk measure since it does not satisfy the subadditivity axiom and hence in general does not respect the diversification principle. Second, VaR is invariant to the extremity of the losses beyond the q-th percentile—the issue, also referred to as "tail risk" problem [126].

To remedy these problems, it was suggested by Artzner et al. in [9] to use the Expected Shortfall (ES) risk measure instead, which was shown to be coherent (see [1, Appendix A]).

Definition 2.20 (Expected Shortfall)**.** *Let $q \in (0,1)$ and $X \in \mathcal{G}$ such that $\mathbb{E}X^+ < \infty$. The* Expected Shortfall (ES) *risk measure at level q for the variable X is*

$$\mathrm{ES}_q(X) := \frac{1}{1-q}\int_q^1 \mathrm{VaR}_u(X)\mathrm{d}u. \tag{2.38}$$

The ES risk measure can be equivalently expressed as the Conditional Value-at-Risk risk measure, introduced in [87, 94, 93] (see [1] for more details as well).

Definition 2.21 (Conditional-Value-at-Risk)**.** *Let $q \in (0,1)$ and $X \in \mathcal{G}$ such that $\mathbb{E}X^+ < \infty$. The* Conditional Value-at-Risk (CVaR) *risk measure at level q for the variable X is*

$$\mathrm{CVaR}_q(X) := \inf_{x\in\mathbb{R}}\left\{x + \frac{\mathbb{E}(X-x)^+}{1-q}\right\}. \tag{2.39}$$

Despite being arguably less intuitive than (2.38), the CVaR representation of ES is convenient, as it allows the ES risk measure to be directly related to the HG risk measure discussed below.

Haezendonck-Goovaerts risk measure

We now review the central risk measure of the dissertation, namely, the Haezendonck-Goovaerts risk measure, based on Orlitz premium principle, which was named after its authors who proposed it in their 1982 paper [60]. Strictly speaking, the Haezendonck-Goovaerts risk measures form a family of functionals, each characterised by a function φ, called a normalised Young function, that governs its behaviour.

Definition 2.22. *A function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is said to be a* normalised Young function *if φ is nonnegative and convex on the interval $[0,\infty)$ and satisfies $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(\infty) = \infty$.*

Before stating the precise definition of the HG measure, it is important to discuss the domain $\mathcal{G}$ on which it is considered. In recent articles [15, 108, 109], the HG risk measure has been studied on the so-called Orlitz spaces which have received growing attention in the context of risk measures [15].

Definition 2.23. *The* Orlicz space $\mathbb{L}^\varphi$ *and* Orlicz heart $\mathbb{L}_0^\varphi$ *of real-valued r.v.s associated with the normalised Young function* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *are defined as*

$$\mathbb{L}^\varphi := \Big\{X : \mathbb{E}(\varphi(c|X|)) < \infty \text{ for some } c > 0\Big\},$$
$$\mathbb{L}_0^\varphi := \Big\{X : \mathbb{E}(\varphi(c|X|)) < \infty \text{ for all } c > 0\Big\}.$$

Depending on the choice of a normalised Young function, the associated Orlitz spaces may coincide. A sufficient condition is given in the following proposition.

Proposition 2.2. *If a normalised Young function* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\limsup_{x\to\infty} \frac{\varphi(2x)}{\varphi(x)} < \infty, \tag{2.40}$$

then $\mathbb{L}^\varphi = \mathbb{L}_0^\varphi$.

Proof. It is clear from Definition 2.23 that $\mathbb{L}^\varphi \supseteq \mathbb{L}_0^\varphi$. It remains to show that $\mathbb{L}^\varphi \subseteq \mathbb{L}_0^\varphi$. Take any $X \in \mathbb{L}^\varphi$. There exists $c_0 > 0$ such that $\mathbb{E}(\varphi(c_0|X|)) < \infty$. From the hypothesis of the proposition, there exist $x_0, M_0 \in \mathbb{R}$ such that $x > x_0$ implies $\varphi(2x)/\varphi(x) \leqslant M_0$. From Definition 2.22, it follows that φ is nondecreasing on the interval $[0, \infty)$. Thus, for $x > x_0$, we have $\varphi(x) \leqslant \varphi(2x) \leqslant M\varphi(x)$. It follows that for all $k \in \mathbb{N}$, there exists $M \in \mathbb{R}$ such that $\varphi(x) \leqslant \varphi(2^k x) \leqslant M\varphi(x)$ for sufficiently large x. Let $c > 0$ be given. If $c \leqslant c_0$, then $\varphi(cx) \leqslant \varphi(c_0 x)$ and, as a result, $\mathbb{E}\varphi(c|X|) \leqslant \mathbb{E}\varphi(c_0|X|) < \infty$. If $c > c_0$, then there exists $M \in \mathbb{R}$, $x_0 \in \mathbb{R}$, $k \in \mathbb{N}$, such that $\varphi(c_0 x) \leqslant \varphi(cx) \leqslant \varphi(2^k c_0 x) \leqslant M\varphi(c_0 x)$ for $x > x_0$. Moreover, the condition $\mathbb{E}\varphi(c_0|X|) < \infty$ implies

$$\int_{(x_0,\infty)} \varphi\left(c_0|x|\right) \mathrm{d}F_X(x) < \infty \quad \text{and} \quad \int_{(-\infty,-x_0)} \varphi\left(c_0|x|\right) \mathrm{d}F_X(x) < \infty.$$

Hence,

$$\int_{(x_0,\infty)} \varphi\left(c|x|\right) \mathrm{d}F_X(x) \leqslant M \int_{(x_0,\infty)} \varphi\left(c_0|x|\right) \mathrm{d}F_X(x) < \infty,$$
$$\int_{(-\infty,-x_0)} \varphi\left(c|x|\right) \mathrm{d}F_X(x) \leqslant M \int_{(-\infty,-x_0)} \varphi\left(c_0|x|\right) \mathrm{d}F_X(x) < \infty.$$

Thus, $\mathbb{E}\varphi(c|X|) < \infty$, and therefore $\mathbb{L}^\varphi = \mathbb{L}_0^\varphi$. □

It is not difficult to find an example of a normalised Young function that satisfies (2.40); for instance, we may take a power function $\varphi(t) = t^\varkappa$, $\varkappa \geqslant 1$.

Having established the preliminaries, we now introduce the HG risk measure.

Definition 2.24 (Haezendonck-Goovaerts risk measure)**.** *Let φ be a normalised Young function, $q \in (0,1)$, and $X \in \mathbb{L}^\varphi$. The* Haezendonck-Goovaerts (HG) risk measure *for the variable X is defined as*

$$\mathrm{HG}_q(X) := \inf_{x\in\mathbb{R}} \left\{x + h(x,q)\right\}, \tag{2.41}$$

where $h = h(x,q)$ is a solution of the equation

$$\mathbb{E}\varphi\left(\frac{(X-x)^+}{h}\right) = 1 - q \tag{2.42}$$

if $\overline{F}_X(x) > 0$ and $h(x,q) = 0$ if $\overline{F}_X(x) = 0$.

The following proposition ensures that the infimum in (2.42) is always attained and is unique under some stricter assumptions.

Proposition 2.3 ([15, Proposition 3 (b, d)])**.** *The minimiser (x_*, h_*) to the equation* (2.41) *of the HG risk measure exists for all $q \in (0,1)$. Moreover, it is unique if the normalised Young function φ is strictly convex and $X \in \mathbb{L}_0^\varphi$.*

What is more, the HG measure satisfies the axioms of coherence (see Axioms M, PH, S, TI) and, as a result, is a coherent and convex risk measure.

Proposition 2.4 ([15, Proposition 3 (e)])**.** *The Haezendonck-Goovaerts risk measure $\mathrm{HG}_q\colon \mathbb{L}^\varphi \to \mathbb{R}$ is coherent.*

As noted in [14], "the class of the Haezendonck measures can be seen as a generalisation of CVaR". Indeed, it easy to see from Definition 2.24, that in the case $\varphi(t) = t$, the expressions (2.41) and (2.42) become

$$\mathrm{HG}_q(X) = \inf_{x\in\mathbb{R}} \left\{ x + \frac{\mathbb{E}\,(X-x)^+}{1-q} \right\},$$

which is equivalent to the definition of CVaR (see Definition2.21).

Depending on the choice of a normalised Young function, the value of the HG risk measure may differ significantly for a given confidence level $q \in (0,1)$. To better illustrate this, we present an example. Let us consider the behaviour of the HG risk measure for a heavy-tailed random risk $X \stackrel{d}{=} \mathcal{P}a_{\{2,2\}}$ under 5 different Young functions $\varphi(t) = t^{\varkappa}$, $\varkappa \in \{1,\ 3/2,\ 2,\ 5/2,\ 3\}$. In the case $\varkappa = 1$, $\mathrm{HG}_q(X) = \mathrm{ES}_q(X)$. As can be seen from Figure 2.5, higher values of the exponent $\varkappa$ result in larger values of the risk measure at a given q, which provides greater flexibility in tailoring the measure to a desired level of risk aversion[3].

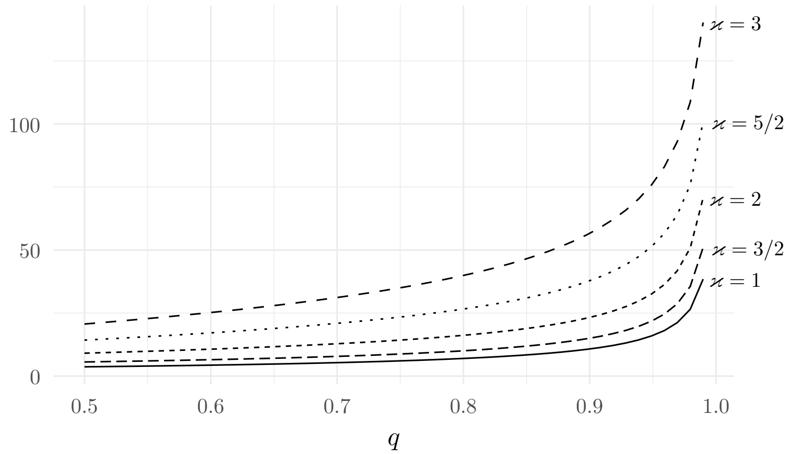

Figure 2.5: The values of HG_q risk measure with $\varphi(t) = t^{\varkappa}$ for a random risk $X \stackrel{d}{=} \mathcal{P}a_{\{2,2\}}$.

[3]The values of $\mathrm{HG}_q(X)$ in the example were estimated by the MC simulations method based on the expressions (5.1), (5.2) given in Theorem 5.1 (see Chapter 5 below).

2.5 Some general results

In this section, we list a couple of useful auxiliary assertions which will be used repeatedly throughout the dissertation. The first one is the so-called *min-max inequality.*

Lemma 2.4 (Min-max inequality)**.** *Let $l \in \mathbb{N}$, $a_i \geqslant 0$, $b_i > 0$, $i = 1, \dots, l$. Then*

$$\min\left\{\frac{a_1}{b_1}, \dots, \frac{a_l}{b_l}\right\} \leqslant \frac{a_1 + \ldots + a_l}{b_1 + \ldots + b_l} \leqslant \max\left\{\frac{a_1}{b_1}, \dots, \frac{a_l}{b_l}\right\}. \tag{2.43}$$

Proof. The assertion of Lemma 2.4 follows immediately by observing that

$$\frac{a_1 + \ldots + a_l}{b_1 + \ldots + b_l} = \sum_{i=1}^{l} \frac{a_i}{b_i} \frac{b_i}{b_1 + \ldots + b_l},$$

where $0 < b_i/(b_1 + \ldots + b_l) < 1$ for $i = 1, \dots, l$. □

The second lemma in this section provides an alternative way to express the expectation of a transformed r.v. Before stating the lemma, we note that a real number m is called the greatest lower bound of a r.v. X if $\mathbb{P}(X \geqslant m) = 1$ and $\mathbb{P}(X \geqslant m + \delta) < 1$ for all $\delta > 0$.

Lemma 2.5 (Alternative expectation formula)**.** *Let ξ be a r.v. with the greatest lower bound m. Let $\varphi : [m, \infty) \to \mathbb{R}$ be a differentiable function with an integrable derivative such that*

$$\int_{[m,\infty)} \left(\int_m^x |\varphi'(t)| \mathrm{d}t \right) \mathrm{d}F_\xi(x) < \infty.$$

Then it holds that

$$\mathbb{E}(\varphi(\xi)) = \varphi(m) + \int_m^\infty \varphi'(t) \overline{F}_\xi(t) \mathrm{d}t.$$

Proof of Lemma 2.5 above can be found in [79]. For more information about alternative expectation formulas, we refer the reader to [86, 102] and the references therein.

3 Tail asymptotics of the product of normal random variables

3.1 Related results

The study of the distribution of the product of normal r.v.s has a long history, with the exact expression for the product of two standard normal r.v.s dating back to the first half of the twentieth century. In particular, it was observed in [123, 133, 33] that for two independent r.v.s $\xi_1, \xi_2 \stackrel{d}{=} \Phi$ the density function of their product is

$$f_{\xi_1 \xi_2}(x) = \frac{1}{\pi} K_0(|x|), \qquad x \in \mathbb{R},$$

where K_0 is the modified Bessel function of the second kind of order zero, i.e.,

$$K_0(x) := \frac{1}{2} \int_0^\infty y^{-1} \exp\left\{ -y - \frac{x^2}{4y} \right\} \mathrm{d}y, \quad x \geqslant 0.$$

Craig in [33] also derived expressions for the moment generating function of the product of two correlated normal variables with non-zero means and arbitrary variances, i.e., $\xi_1 \xi_2$, where $\xi_1 \stackrel{d}{=} \Phi_{\{\mu_1, \sigma_1\}}$, $\xi_2 \stackrel{d}{=} \Phi_{\{\mu_2, \sigma_2\}}$, $\rho = \operatorname{Corr}(\xi_1, \xi_2) \in (-1, 1)$. Additionally, the expression was obtained for the density function of the product of such r.v.s, when $\rho = 0$, although it involves integrals or infinite series of modified Bessel functions. The results obtained by Craig were later extended and generalised in a number of subsequent papers. For instance, Haldane in [61] derived a cumulant generating function, the first four moments and the first four cumulants of the products of normal random variables. Aronian in [7] showed that the distribution of product of two normal random variables can be approximated by a normal distribution when the ratios μ_1/σ_1, μ_2/σ_2 of the normally distributed multipliers are large. In [8] and [99] the expressions for the product distribution of two normal r.v.s were provided in the special cases of parameters and dependence structures. Almost 80 years later, the exact expressions for two correlated zero-mean normal random variables were obtained. In [82, 54] it was shown that if $\xi_1 \stackrel{d}{=} \Phi_{\{0,\sigma_1\}}$, $\xi_2 \stackrel{d}{=} \Phi_{\{0,\sigma_2\}}$ with $\rho \in (-1, 1)$, then

$$f_{\xi_1\xi_2}(x) = \frac{1}{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{\rho x}{\sigma_1\sigma_2(1-\rho^2)}\right\} K_0\left(\frac{|x|}{\sigma_1\sigma_2(1-\rho^2)}\right),$$

$x \in \mathbb{R}$. At similar times the exact formula for the distribution of two correlated normal r.v.s with non-zero means and arbitrary variances was obtained in [34]; however, the expression obtained is considerably more complex due to the presence of infinite sums of modified Bessel functions of the second kind. More recently, Gaunt in [55] revisited the problem of finding the expression of density for the product of correlated zero-mean normal random variables and proposed a new proof technique based on the so-called Stein's method.

Along with the exact formulas for the product distribution, the asymptotics of the tail distribution is often of interest. One of the results in this direction was obtained by Arendarczyk and Dębicki in [6] in the context of Gaussian processes and was applied for the time distribution with Weibullian-type tails. The result, as we will see shortly, applies to normal distributions as well.

Theorem 3.1 (See [6, Lemma 2.1]). *Let ξ_1 and ξ_2 be two independent, nonnegative r.v.s, such that,*

$$\overline{F}_{\xi_1}(x) \underset{x\to\infty}{\sim} A_1 x^{\gamma_1} \exp\{-\beta_1 x^{\alpha_1}\},\ \ \overline{F}_{\xi_2}(x) \underset{x\to\infty}{\sim} A_2 x^{\gamma_2} \exp\{-\beta_2 x^{\alpha_2}\},$$

where $A_i > 0$, $\gamma_i \in \mathbb{R}$, $\beta_i > 0$, $\alpha_i > 0$, $i = 1, 2$. Then

$$\overline{F}_{\xi_1\xi_2}(x) \underset{x\to\infty}{\sim} A x^{\gamma} \exp\{-\beta x^{\alpha}\},$$

where

$$\alpha = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2},$$

$$\begin{aligned}\beta &= \beta_1^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \beta_2^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \left(\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} + \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \right), \\ \gamma &= \frac{\alpha_1\alpha_2 + 2\alpha_1\gamma_2 + 2\alpha_2\gamma_1}{2(\alpha_1+\alpha_2)}, \\ A &= \sqrt{2\pi}\, \frac{A_1 A_2}{\sqrt{\alpha_1+\alpha_2}}\, (\alpha_1\beta_1)^{\frac{\alpha_2-2\gamma_1+2\gamma_2}{2(\alpha_1+\alpha_2)}}\, (\alpha_2\beta_2)^{\frac{\alpha_1-2\gamma_2+2\gamma_1}{2(\alpha_1+\alpha_2)}}\,.\end{aligned}$$

In particular, this theorem directly implies that if ξ_1, ξ_2, ξ_3 are i.i.d. standard normal r.v.s, then

$$\overline{F}_{\xi_1\xi_2}(x) \underset{x\to\infty}{\sim} \frac{1}{\sqrt{2\pi}}\, x^{-\frac{1}{2}} \mathrm{e}^{-x}, \;\; \overline{F}_{\xi_1\xi_2\xi_3}(x) \underset{x\to\infty}{\sim} \frac{2}{\sqrt{6\pi}}\, x^{-\frac{1}{3}} \mathrm{e}^{-\frac{3}{2}x^{2/3}}. \tag{3.1}$$

Let us describe the derivation of the above formulas. At first, suppose $n = 2$. R.v.s ξ_1, ξ_2 are distributed according to the symmetric law Φ. Therefore,

$$\overline{F}_{\xi_1\xi_2}(x) = \mathbb{P}(\xi_1\xi_2 > x) = 2\mathbb{P}(\xi_1^+\xi_2^+ > x) = 2\overline{F}_{\xi_1^+\xi_2^+}(x), \; x > 0. \tag{3.2}$$

For $x > 0$, by partial integration, we obtain the following bounds (see also [46, Section 7.1]):

$$\frac{1}{\sqrt{2\pi}}\Big(\frac{1}{x} - \frac{1}{x^3}\Big)\mathrm{e}^{-x^2/2} \leqslant \overline{F}_{\xi_1}(x) \leqslant \frac{1}{\sqrt{2\pi}}\,\frac{1}{x}\,\mathrm{e}^{-x^2/2}. \tag{3.3}$$

Indeed, observe that

$$\begin{aligned}\overline{F}_{\xi_1} &= \frac{1}{\sqrt{2\pi}} \int_x^\infty \mathrm{e}^{-y^2/2} \mathrm{d}y = -\frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{1}{y} \mathrm{d}\mathrm{e}^{-y^2/2} \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{x}\mathrm{e}^{-x^2/2} - \int_x^\infty \frac{1}{y^3}\mathrm{e}^{-y^2/2}\mathrm{d}y\right)\end{aligned} \tag{3.4}$$

implying the upper bound in (3.3). Integrating by parts the integral in (3.4) we get

$$\overline{F}_{\xi_1} = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{x}\mathrm{e}^{-x^2/2} - \frac{1}{x^3}\mathrm{e}^{-x^2/2} + \int_x^\infty \frac{3}{y^4}\mathrm{e}^{-y^2/2}\mathrm{d}y\right)$$

and the lower bound in (3.3) follows.

To derive the asymptotic formula for $\overline{F}_{\xi_1^+\xi_2^+}(x)$ we apply Theorem 3.1

with

$$\alpha_1 = \alpha_2 = 2,\ \beta_1 = \beta_2 = \frac{1}{2},\ \gamma_1 = \gamma_2 = -1,\ A_1 = A_2 = \frac{1}{\sqrt{2\pi}},$$

since (3.3) implies

$$\overline{F}_{\xi_1^+}(x) \underset{x\to\infty}{\sim} \frac{1}{\sqrt{2\pi}} x^{-1} \mathrm{e}^{-x^2/2}. \tag{3.5}$$

After calculating the values for α, β, γ, A we obtain

$$\overline{F}_{\xi_1^+\xi_2^+}(x) \underset{x\to\infty}{\sim} \frac{1}{2\sqrt{2\pi}} x^{-1/2} \mathrm{e}^{-x}. \tag{3.6}$$

The asymptotics in (3.6) together with the equality (3.2) implies the first formula in (3.1).

In the case of $n = 3$, similarly to the formula (3.2), it holds that

$$\overline{F}_{\xi_1\xi_2\xi_3}(x) = 4\overline{F}_{\xi_1^+\xi_2^+\xi_3^+}(x),\ x > 0. \tag{3.7}$$

To derive the asymptotics for $\overline{F}_{\xi_1^+\xi_2^+\xi_3^+}(x)$ we can apply Theorem 3.1 again by using the asymptotic formulas (3.5) and (3.6). According to these relations, parameters in Theorem 3.1 are the following:

$$\begin{gathered}\alpha_1 = 1,\ \beta_1 = 1,\ \gamma_1 = -\frac{1}{2},\ A_1 = \frac{1}{2\sqrt{2\pi}},\\ \alpha_2 = 2,\ \beta_2 = \frac{1}{2},\ \gamma_2 = -1,\ A_2 = \frac{1}{\sqrt{2\pi}}.\end{gathered}$$

Due to the formulas of Theorem 3.1, we derive

$$\overline{F}_{\xi_1^+\xi_2^+\xi_3^+}(x) \underset{x\to\infty}{\sim} A x^{\gamma} \mathrm{e}^{-\beta x^{\alpha}} \tag{3.8}$$

with $\alpha = 2/3, \beta = 3/2, \gamma = -1/3, A = 1/2\sqrt{6\pi}$. To obtain the second formula in (3.1) it is enough to use the formula (3.7).

By generalising the above discussion and Theorem 3.1, we state the following result.

Proposition 3.1. *Let $\xi_1, \xi_2, \ldots, \xi_n$, $n \in \mathbb{N}$, be i.i.d. r.v.s, such that for* *each* k, $\xi_k \stackrel{d}{=} \Phi$ *and let* $\Pi_n := \prod_{k=1}^n \xi_k$. *Then*

$$\overline{F}_{\Pi_n}(x) \underset{x\to\infty}{\sim} \frac{2^{(n/2)-1}}{\sqrt{\pi n}} x^{-\frac{1}{n}} \exp\left\{-\frac{n}{2}x^{\frac{2}{n}}\right\}. \tag{3.9}$$

Proof. First, observe that by symmetry of standard normal distribution we can write for $x > 0$, using the formula of total probability,

$$\overline{F}_{\Pi_n}(x) = C(n)\mathbb{P}(\xi_1^+ \cdots \xi_n^+ > x) = C(n)\overline{F}_{\xi_1^+\cdots\xi_n^+}(x), \tag{3.10}$$

where $C(n) = 2^{n-1}$ and represents the number of $\{+,-\}$ sign assignments to an ordered list of length n with an even total number of $-$ signs.

Hence, it suffices to show that

$$\overline{F}_{\xi_1^+\cdots\xi_n^+}(x) \underset{x\to\infty}{\sim} \frac{2^{-n/2}}{\sqrt{\pi n}} x^{-\frac{1}{n}} \exp\left\{-\frac{n}{2}x^{\frac{2}{n}}\right\}. \tag{3.11}$$

We proceed by induction on n. We have already shown that the formula (3.11) holds in the cases $n = 1, 2, 3$ (see (3.4), (3.5), (3.8)). Assume that the formula 3.11 holds for $n = m$, $m \geqslant 2$, i.e.,

$$\overline{F}_{\xi_1^+\cdots\xi_m^+}(x) \underset{x\to\infty}{\sim} \frac{2^{-m/2}}{\sqrt{\pi m}} x^{-\frac{1}{m}} \exp\left\{-\frac{m}{2}x^{\frac{2}{m}}\right\}.$$

In the case $n = m + 1$ the we can apply Theorem 3.1 for the r.v.s $\Pi_n^* := \xi_1^+ \cdots \xi_m^+$ and ξ_{m+1}^+ with the collections of parameters

$$\left\{\alpha_1 = \frac{2}{m},\ \beta_1 = \frac{m}{2},\ \gamma_1 = -\frac{1}{m},\ A_1 = \frac{2^{-m/2}}{\sqrt{\pi m}}\right\}$$

and

$$\left\{\alpha_2 = 2,\ \beta_2 = \frac{1}{2},\ \gamma_2 = -1,\ A_2 = \frac{1}{2\sqrt{\pi}}\right\}$$

respectively. After some algebra, we obtain

$$\overline{F}_{\xi_1^+\cdots\xi_{m+1}^+}(x) = Ax^{\gamma}\mathrm{e}^{-\beta x^{\alpha}}$$

with

$$\alpha = \frac{2}{1+m}, \quad \beta = \frac{m+1}{2}, \quad \gamma = -\frac{1}{m+1}, \quad A = \frac{2^{-(m+1)/2}}{\sqrt{\pi(m+1)}}.$$

This completes the proof. $\square$

It turns out that Proposition 3.1 can be simply extended to the not necessarily standard normal variables with zero means.

Corollary 3.1. *Let $\xi_1, \xi_2, \ldots, \xi_n$, $n \in \mathbb{N}$, be independent r.v.s, such that for each k, $\xi_k \overset{d}{=} \Phi_{\{0,\sigma_k\}}$ and let $\widehat{\Pi}_n := \prod_{k=1}^n \xi_k$. Then*

$$\overline{F}_{\widehat{\Pi}_n}(x) \underset{x\to\infty}{\sim} \frac{2^{(n/2)-1}(\sigma^{(n)})^{1/n}}{\sqrt{\pi n}} x^{-\frac{1}{n}} \exp\left\{-\frac{n(\sigma^{(n)})^{-2/n}}{2} x^{\frac{2}{n}}\right\}, \quad (3.12)$$

where $\sigma^{(n)} := \prod_{k=1}^n \sigma_k$.

Remark 3.1. *The assertion of Corollary 3.1 follows from Proposition 3.1 immediately, because for each k, r.v. $\xi_k = \widehat{\xi}_k/\sigma_k$ is distributed according to Φ and*

$$\overline{F}_{\widehat{\Pi}_n}(x) = \mathbb{P}\left(\widehat{\Pi}_n > x\right) = \mathbb{P}\left(\Pi_n > \frac{x}{\sigma^{(n)}}\right) = \overline{F}_{\Pi_n}\left(\frac{x}{\sigma^{(n)}}\right).$$

An important consequence of Proposition 3.1 and Corollary 3.1 is that depending on the number of multipliers in the product of normal random variables the resulting product is either light or heavy tailed. This observation is formalised in the following lemma.

Lemma 3.1. *Let $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, be independent r.v.s, such that $\xi_k \overset{d}{=} \Phi_{\{0,\sigma_k\}}$, $k = 1, \ldots, n$ and $\widehat{\Pi}_n := \prod_{k=1}^n \xi_k$. The product $\widehat{\Pi}_n$ is light-tailed if $n \in \{1, 2\}$ and heavy-tailed if $n > 2$.*

Proof. Using Lemma 2.1 and (3.12), we have

$$\begin{aligned}\limsup_{x\to\infty} \frac{\overline{F}_{\widehat{\Pi}_n}(x)}{\mathrm{e}^{-\delta x}} = \lim_{x\to\infty} & \frac{2^{(n/2)-1}(\sigma^{(n)})^{1/n}}{\sqrt{\pi n}} x^{-\frac{1}{n}} \\ & \times \exp\left\{-\frac{n(\sigma^{(n)})^{-2/n}}{2} x^{\frac{2}{n}} + \delta x\right\}.\end{aligned}$$

For $n = 1$, $\limsup_{x\to\infty} \overline{F}_{\widehat{\Pi}_n}(x)\mathrm{e}^{\delta x} < \infty$ for all $\delta > 0$ (see also Example 2.1 for an alternative proof), while for $n = 2$ the same holds for $\delta \in (0, (\sigma^{(2)})^{-1})$. Thus, $\overline{F}_{\widehat{\Pi}_n}$ is a light-tailed distribution for $n \in (1, 2)$. Let us consider the case $n > 2$. It is clear that the exponent in the expression above converge to infinity for all $\delta > 0$ and, as a result, the $\limsup_{x\to\infty} \overline{F}_{\widehat{\Pi}_n}(x)\mathrm{e}^{\delta x} = \infty$. The proof is complete. $\square$

3.2 Main results

In this section, we derive a more precise formula for the tail of the product distribution of normal r.v.s. The following two statements are the main results regarding the tail asymptotics of the product distribution of normal r.v.s. Theorem 3.2 establishes the asymptotic formula with the first leading-order term for the product of standard normal r.v.s, while Corollary 3.2 generalises Theorem 3.2 to the case of not necessarily identically distributed centred normal r.v.s with arbitrary variances.

Theorem 3.2 (See [73, Theorem 2.2]). *Let $\xi_1, \xi_2, \ldots, \xi_n$, $n \in \mathbb{N}$, be i.i.d. r.v.s such that for each k, $\xi_k \overset{d}{=} \Phi$ and let $\Pi_n := \prod_{k=1}^{n} \xi_k$. Then*

$$\overline{F}_{\Pi_n}(x) = \frac{2^{(n/2)-1}}{\sqrt{\pi n}}\, x^{-\frac{1}{n}} \exp\left\{-\frac{n}{2}\, x^{\frac{2}{n}}\right\} \left(1 + O_n\left(x^{-\frac{2}{n}}\right)\right) \tag{3.13}$$

as $x \to \infty$, where the implicit constant in O_n depends on n.

Applying the same reasoning as in Remark 3.1 we obtain the following corollary.

Corollary 3.2. *Let $\xi_1, \xi_2, \ldots, \xi_n$, $n \in \mathbb{N}$, be independent r.v.s such that for each k, $\xi_k \overset{d}{=} \Phi_{\{0,\sigma_k\}}$, and let $\widehat{\Pi}_n := \prod_{k=1}^{n} \xi_k$. Then*

$$\begin{aligned}\overline{F}_{\widehat{\Pi}_n}(x) = \frac{2^{(n/2)-1}(\sigma^{(n)})^{1/n}}{\sqrt{\pi n}} x^{-\frac{1}{n}} \\ \times \exp\left\{-\frac{n(\sigma^{(n)})^{-2/n}}{2}\, x^{\frac{2}{n}}\right\} \left(1 + O_n\left(x^{-\frac{2}{n}}\right)\right)\end{aligned}$$

as $x \to \infty$, where $\sigma^{(n)} := \prod_{k=1}^{n} \sigma_k$ and the implicit constant in O_n depends on n and $\sigma^{(n)}$.

3.3 Proofs of the main results

3.3.1 Auxiliary assertions

The asymptotic formulas presented in Theorem 3.1 were obtained using the saddlepoint approximation method described in detail in [22]. We rely on this method indirectly as well. To prove Theorem 3.2, we repeatedly use an auxiliary lemma stated below, which itself is obtained by applying the saddlepoint method to a real integral of a particular form.

Lemma 3.2 (See [124, Theorem 1 in Section 2]). *Let h and g be two real functions defined on an interval $[a, b)$, $-\infty < a < b \leqslant \infty$, such that:*

$$\text{(i)} \quad \begin{aligned} h(z) &= h(a) + \sum_{k=0}^{N} a_k (z-a)^{k+\mu} + o\left((z-a)^{N+\mu}\right), \\ g(z) &= \sum_{k=0}^{N} b_k (z-a)^{k+\alpha-1} + o\left((z-a)^{N+\alpha-1}\right), \\ h'(z) &= \sum_{k=1}^{N} (k+\mu) a_k (z-a)^{k+\mu-1} + o\left((z-a)^{N+\mu-1}\right) \end{aligned}$$

as $z \downarrow a$ for all $N \geqslant 1$, where $a_0 \neq 0$, $b_0 \neq 0$, $\mu > 0$ and $\alpha > 0$;

$$\text{(ii)} \quad h(z) > h(a) \ \textit{for}\ z \in (a, b), \textit{ and}$$

$$\inf_{z \in [a+\delta, b)} (h(z) - h(a)) > 0 \ \textit{for each}\ \delta > 0;$$

(iii) *h' and g are continuous in a neighbourhood of point a.*

If the integral $\int_a^b g(z) \mathrm{e}^{-xh(z)} \mathrm{d}z$ converges absolutely for all sufficiently large x, then

$$\int_a^b g(z) \mathrm{e}^{-xh(z)} \mathrm{d}z = \mathrm{e}^{-xh(a)} \left(\sum_{k=0}^{N} \Gamma\left(\frac{k+\alpha}{\mu}\right) d_k x^{-\frac{k+\alpha}{\mu}} + O\left(x^{-\frac{N+\alpha+1}{\mu}}\right)\right)$$

as $x \to \infty$ for all $N \in \mathbb{N}$, where Γ denotes the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \mathrm{d}t, \quad x > 0,$$

while coefficients d_k are expressible in terms of a_k and b_k.

Wong in [124] also provided the explicit forms of the first three coefficients:

$$\begin{aligned} d_0 &= \frac{b_0}{\mu a_0^{\alpha/\mu}}, \quad d_1 = \left(\frac{b_1}{\mu} - \frac{(\alpha+1) a_1 b_0}{\mu^2 a_0}\right) \frac{1}{a_0^{(\alpha+1)/\mu}}, \\ d_2 &= \left(\frac{b_2}{\mu} - \frac{(\alpha+2) a_1 b_1}{\mu^2 a_0}\right. \end{aligned}$$

$$+\left(\left((\alpha+\mu+2)a_1^2-2\mu a_0 a_2\right)\frac{(\alpha+2)b_0}{2\mu^3 a_0^2}\right)\frac{1}{a_0^{(\alpha+2)/\mu}}.$$

3.3.2 Main proof

Proof of Theorem 3.2. We use induction on n. The estimate (3.3) implies the assertion of Theorem 5.11 in the case $n=1$.

Suppose $n=2$. By symmetry of the distribution Φ, for $x>0$ we have

$$\begin{aligned}
\overline{F}_{\xi_1\xi_2}(x) &= 2\int_0^\infty \overline{F}_{\xi_1}\left(\frac{x}{y}\right)\varphi(y)\mathrm{d}y \\
&= \frac{2}{\sqrt{2\pi}}\left(\int_0^{2x^{1/2}}+\int_{2x^{1/2}}^\infty\right)\overline{F}_{\xi_1}\left(\frac{x}{y}\right)\mathrm{e}^{-y^2/2}\mathrm{d}y \\
&=: I_1+I_2. \qquad (3.14)
\end{aligned}$$

Obviously,

$$\begin{aligned}
I_2 &\leqslant \frac{1}{\sqrt{2\pi}}\int_{2x^{1/2}}^\infty \mathrm{e}^{-y^2/2}\mathrm{d}y = \frac{1}{\sqrt{2\pi}}\int_{2x}^\infty (2u)^{-1/2}\mathrm{e}^{-u}\mathrm{d}u \\
&\leqslant C_{11}x^{-1/2}\mathrm{e}^{-2x} \qquad (3.15)
\end{aligned}$$

for some positive constant C_{11}.

For the integral I_1, due to the first inequality in (3.3), we have

$$\begin{aligned}
I_1 &\leqslant \frac{1}{\pi}\int_0^{2x^{1/2}}\frac{y}{x}\,\mathrm{e}^{-\frac{1}{2}(\frac{x^2}{y^2}+y^2)}\mathrm{d}y = \frac{1}{2\pi}\int_0^4 \mathrm{e}^{-\frac{x}{2}(u+\frac{1}{u})}\mathrm{d}u \\
&= \frac{1}{2\pi}\left(\int_0^1+\int_1^4\right)\mathrm{e}^{-\frac{x}{2}(u+\frac{1}{u})}\mathrm{d}u =: I_{11}+I_{12}, \qquad (3.16)
\end{aligned}$$

where the variable change $y=\sqrt{xu}$ was used in the second step.

Using Lemma 3.2 for the integral I_{12} (with $\mu=2$, $\alpha=1$, $a_0=b_0=1$, $a_1=-1$, $b_1=0$), we get

$$\begin{aligned}
I_{12} &= \frac{\mathrm{e}^{-x}}{2\pi}\left(\Gamma\left(\frac{1}{2}\right)\left(\frac{2}{x}\right)^{1/2}\frac{1}{2}+\Gamma(1)\,\frac{2}{x}\,\frac{1}{2}+O\left(x^{-3/2}\right)\right) \\
&= \frac{\mathrm{e}^{-x}}{2\pi}\left(\sqrt{\frac{\pi}{2}}\,x^{-1/2}+x^{-1}+O\left(x^{-3/2}\right)\right) \qquad (3.17)
\end{aligned}$$

as $x \to \infty$, because $u + \frac{1}{u}\Big|_{u=1} = 2$ and

$$u + \frac{1}{u} = 2 + \sum_{k=0}^{N} (-1)^{k+2}(u-1)^{k+2} + o\left((u-1)^{N+2}\right), \quad u \to 1, \tag{3.18}$$

for all $N \geqslant 1$ due to the Taylor formula.

In a similar way, using Lemma 3.2 (with $\mu = 2$, $\alpha = 1$, $a_0 = b_0 = 1$, $a_1 = -1$, $b_1 = -2$), we derive that

$$\begin{aligned} I_{11} = \frac{1}{2\pi}\int_0^1 \mathrm{e}^{-\frac{x}{2}(u+\frac{1}{u})}\mathrm{d}u &= \frac{1}{2\pi}\int_1^\infty \mathrm{e}^{-\frac{x}{2}(v+\frac{1}{v})}\frac{1}{v^2}\,\mathrm{d}v \\ &= \frac{\mathrm{e}^{-x}}{2\pi}\left(\Gamma\left(\frac{1}{2}\right)\left(\frac{2}{x}\right)^{1/2}\frac{1}{2} + \Gamma(1)\frac{2}{x}\left(-\frac{1}{2}\right) + O\left(x^{-3/2}\right)\right) \\ &= \frac{\mathrm{e}^{-x}}{2\pi}\left(\sqrt{\frac{\pi}{2}}\,x^{-1/2} - x^{-1} + O\left(x^{-3/2}\right)\right) \end{aligned} \tag{3.19}$$

as $x \to \infty$, because of (3.18) and the following decomposition:

$$\frac{1}{v^2} = \sum_{k=0}^{N}(-1)^k(k+1)(v-1)^k + o((v-1)^N), \quad v \to 1,$$

according to the Taylor formula.

The estimates (3.16), (3.17) and (3.19) imply that for some $C_{12} > 0$ and all sufficiently large x,

$$I_1 \leqslant \frac{1}{\sqrt{2\pi}}x^{-1/2}\mathrm{e}^{-x}(1 + C_{12}x^{-1}). \tag{3.20}$$

Substituting the derived estimates (3.15) and (3.20) into (3.14), we obtain

$$\overline{F}_{\xi_1\xi_2}(x) \leqslant \frac{1}{\sqrt{2\pi}}x^{-1/2}\mathrm{e}^{-x}\left(1 + C_2x^{-1}\right) \tag{3.21}$$

with a positive constant C_2 for all sufficiently large x.

Now let us consider the lower bound for $\overline{F}_{\xi_1\xi_2}(x)$. The equality (3.14) and the lower bound in (3.3) imply

$$\overline{F}_{\xi_1\xi_2}(x) \geqslant \frac{2}{\sqrt{2\pi}}\int_0^{2x^{1/2}} \overline{F}_{\xi_1}\left(\frac{x}{y}\right)\mathrm{e}^{-y^2/2}\mathrm{d}y$$

$$
\begin{aligned}
&\geqslant \frac{1}{\pi}\int_0^{2x^{1/2}} \frac{y}{x}\left(1-\frac{y^2}{x^2}\right) \mathrm{e}^{-\frac{1}{2}(y^2+\frac{x^2}{y^2})}\mathrm{d}y \\
&\geqslant \frac{1}{2\pi}\int_0^{4}\left(1-\frac{u}{x}\right) \mathrm{e}^{-\frac{x}{2}(u+\frac{1}{u})}\mathrm{d}u \\
&\geqslant \frac{1}{2\pi}\left(1-\frac{4}{x}\right)\int_0^{4} \mathrm{e}^{-\frac{x}{2}(u+\frac{1}{u})}\mathrm{d}u \\
&= \left(1-\frac{4}{x}\right)(I_{11}+I_{12}).
\end{aligned}
$$

Thus, due to the estimates (3.17) and (3.19),

$$
\overline{F}_{\xi_1\xi_2}(x) \geqslant \frac{1}{\sqrt{2\pi}}\, x^{-1/2}\mathrm{e}^{-x}\left(1-C_3x^{-1}\right) \tag{3.22}
$$

for a positive constant C_3 and all sufficiently large x. The inequalities (3.21) and (3.22) imply the assertion of the theorem in the case $n=2$.

Let us suppose now that the formula (3.13) holds for $n=m$, $m\geqslant 2$, i.e., let

$$
\overline{F}_{\Pi_m}(x) = \frac{2^{(m/2)-1}}{\sqrt{\pi m}}\, x^{-1/m}\exp\left\{-\frac{m}{2}\, x^{2/m}\right\}\left(1+O_m(x^{-2/m})\right) \tag{3.23}
$$

as $x\to\infty$. Obviously,

$$
\begin{aligned}
\overline{F}_{\Pi_{m+1}}(x) &= \frac{2}{\sqrt{2\pi}}\int_0^{\infty}\overline{F}_{\Pi_m}\left(\frac{x}{y}\right)\mathrm{e}^{-y^2/2}\mathrm{d}y \\
&= \frac{2}{\sqrt{2\pi}}\left(\int_0^{(m+1)x^{1/(m+1)}}+\int_{(m+1)x^{1/(m+1)}}^{\infty}\right)\overline{F}_{\Pi_m}\left(\frac{x}{y}\right)\mathrm{e}^{-y^2/2}\mathrm{d}y \\
&=: J_1+J_2.
\end{aligned} \tag{3.24}
$$

According to (3.3),

$$
\begin{aligned}
J_2 &\leqslant \frac{2}{\sqrt{2\pi}}\int_{(m+1)x^{1/(m+1)}}^{\infty}\mathrm{e}^{-y^2/2}\mathrm{d}y \\
&\leqslant C_{4m}x^{-\frac{1}{m+1}}\exp\left\{-\frac{(m+1)^2}{2}\, x^{2/(m+1)}\right\},
\end{aligned} \tag{3.25}
$$

with a positive quantity C_{4m} depending only on m, for all sufficiently large x.

Using the induction hypothesis (3.23) and the variable change $y=$

$x^{1/(m+1)}u^{1/2}$, we get

$$\begin{aligned}
J_1 \;&\leqslant\; \frac{2^{(m-1)/2}}{\pi\sqrt{m}} \int_0^{(m+1)x^{1/(m+1)}} \left(\frac{y}{x}\right)^{1/m} \\
&\qquad \times\, \exp\left\{-\frac{1}{2}\left(m\left(\frac{x}{y}\right)^{2/m}+y^2\right)\right\}\left(1+C_{5m}\left(\frac{y}{x}\right)^{2/m}\right)\mathrm{d}y \\
&= \frac{2^{(m-3)/2}}{\pi\sqrt{m}} \int_0^{(m+1)^2} u^{-(m-1)/(2m)} \exp\left\{-\frac{1}{2}\,x^{2/(m+1)}\left(mu^{-1/m}+u\right)\right\} \\
&\qquad\quad \times\, \left(1+C_{5m}\left(x^{-2/(m+1)}u^{1/m}\right)\right)\mathrm{d}u \\
&\leqslant\; \frac{2^{(m-3)/2}}{\pi\sqrt{m}} \left(1+C_{6m}\left(x^{-2/(m+1)}\right)\right) \\
&\qquad \times \int_0^{(m+1)^2} u^{-(m-1)/(2m)} \exp\left\{-\frac{1}{2}\,x^{2/(m+1)}\left(mu^{-1/m}+u\right)\right\}\mathrm{d}u \\
&= \frac{2^{(m-3)/2}}{\pi\sqrt{m}} \left(1+C_{6m}\left(x^{-2/(m+1)}\right)\right)(J_{11}+J_{12}), \qquad (3.26)
\end{aligned}$$

where C_{5m} and C_{6m} are positive quantities that depend only on m, and

$$\begin{aligned}
J_{11} \;&:=\; \int_0^1 u^{-(m-1)/(2m)} \exp\left\{-\frac{1}{2}\,x^{2/(m+1)}\left(mu^{-1/m}+u\right)\right\}\mathrm{d}u, \\
J_{12} \;&:=\; \int_1^{(m+1)^2} u^{-(m-1)/(2m)} \exp\left\{-\frac{1}{2}\,x^{2/(m+1)}\left(mu^{-1/m}+u\right)\right\}\mathrm{d}u
\end{aligned}$$

for all sufficiently large x.

Using Lemma 3.2 (with $\mu = 2$, $\alpha = 1$, $a_0 = (m+1)/(2m)$, $a_1 = -(m+1)(2m+1)/(6m^2)$, $b_0 = 1$, $b_1 = -(m-1)/(2m)$) we obtain

$$\begin{aligned}
J_{12} \;&=\; \exp\left\{-\frac{m+1}{2}\,x^{2/(m+1)}\right\}\left(\Gamma\left(\frac{1}{2}\right)\sqrt{\frac{m}{2(m+1)}}\left(\frac{x^{2/(m+1)}}{2}\right)^{-1/2}\right. \\
&\qquad\qquad \left. +\,\Gamma(1)\,\frac{m+5}{2(m+1)}\left(\frac{x^{2/(m+1)}}{2}\right)^{-1} + O_m\left(x^{-3/(m+1)}\right)\right) \\
&= \exp\left\{-\frac{m+1}{2}\,x^{2/(m+1)}\right\}\left(\sqrt{\frac{\pi m}{m+1}}\,x^{-1/(m+1)}\right. \\
&\qquad\qquad \left. +\,\frac{m+5}{2(m+1)}\,x^{-2/(m+1)} + O_m\left(x^{-3/(m+1)}\right)\right) \qquad (3.27)
\end{aligned}$$

as $x \to \infty$, because $mu^{-1/m} + u\,\Big|_{u=1} = m+1$, and

$$
\begin{aligned}
mu^{-1/m} + u \;&=\; m + 1 + \sum_{k=0}^{N} (-1)^k \frac{\prod_{l=1}^{k+1}(1+lm)}{(k+2)!m^{k+1}} (u-1)^{k+2} \\
&\quad + o\left((u-1)^{N+2}\right), \\
u^{-(m-1)/(2m)} \;&=\; 1 + \sum_{k=1}^{N} \frac{(-1)^k \prod_{l=1}^{k}((2l-1)m-1)}{k!(2m)^k} (u-1)^k \\
&\quad + o\left((u-1)^{N}\right)
\end{aligned}
$$

for each $N \geqslant 1$ as $u \to 1$ due to the Taylor formula.

Concerning the term J_{11}, after the variable change $u = 1/v$ we get

$$
J_{11} = \int_1^{\infty} v^{-(3m+1)/(2m)} \exp\left\{ -\frac{x^{2/(m+1)}}{2} \left(mv^{1/m} + v^{-1} \right) \right\} \mathrm{d}v.
$$

Therefore, using Lemma 3.2 again (with $\mu = 2$, $\alpha = 1$, $a_0 = (m+1)/(2m)$, $a_1 = -(4m-1)(m+1)/(6m^2)$, $b_0 = 1$, $b_1 = -(3m+1)/(2m)$), similarly as in (3.27), we get that

$$
\begin{aligned}
J_{11} \;&=\; \exp\left\{ -\frac{m+1}{2} x^{\frac{2}{m+1}} \right\} \Bigg(\Gamma\left(\frac{1}{2}\right) \sqrt{\frac{m}{2(m+1)}} \left(\frac{x^{2/(m+1)}}{2} \right)^{-1/2} \\
&\qquad + \Gamma(1) \frac{m+5}{6(m+1)} \left(\frac{x^{2/(m+1)}}{2} \right)^{-1} + O_m\left(x^{-3/(m+1)} \right) \Bigg) \\
&=\; \exp\left\{ -\frac{m+1}{2} x^{2/(m+1)} \right\} \Bigg(\sqrt{\frac{\pi m}{m+1}} x^{-1/(m+1)} \\
&\qquad - \frac{m+5}{2(m+1)} x^{-2/(m+1)} + O_m\left(x^{-3/(m+1)} \right) \Bigg)
\end{aligned} \tag{3.28}
$$

as $x \to \infty$, because $mv^{1/m} + v^{-1} \Big|_{v=1} = m+1$ and

$$
\begin{aligned}
mv^{1/m} + v^{-1} \;&=\; m + 1 + \sum_{k=0}^{N} \left(\frac{\prod_{l=1}^{k+1}(1-lm)}{(k+2)!m^{k+1}} + (-1)^k \right) (v-1)^{k+2} \\
&\quad + o\left((v-1)^{N+2}\right), \\
v^{-(3m+1)/(2m)} \;&=\; 1 + \sum_{k=1}^{N} \frac{(-1)^k \prod_{l=1}^{k}((2l+1)m+1)}{k!(2m)^k} (v-1)^k \\
&\quad + o\left((v-1)^{N}\right)
\end{aligned}
$$

for each $N \geqslant 1$ as $v \to 1$. The equations (3.24)–(3.28) imply that

$$
\begin{aligned}
\overline{F}_{\Pi_{m+1}}(x) \leqslant\ & \frac{2^{(m+1)/2-1}}{\sqrt{\pi(m+1)}}\, x^{-1/(m+1)} \exp\left\{-\frac{m+1}{2}\, x^{2/(m+1)}\right\} \\
& \times \left(1 + C_{7m} x^{-2/m+1}\right) \qquad (3.29)
\end{aligned}
$$

with a positive quantity C_{7m} depending only on m, for all sufficiently large x.

On the other hand, by the hypothesis (3.23), similarly as in the derivation of (3.26), we get that

$$
\begin{aligned}
\overline{F}_{\Pi_{m+1}}(x) \geqslant\ & \frac{2^{(m+1)/2-2}}{\pi\sqrt{m}} \left(1 - C_{8m} x^{-2/(m+1)}\right) \\
& \times \int_0^{(m+1)^2} u^{-\frac{m-1}{2m}} \exp\left\{-\frac{x^{2/(m+1)}}{2}\left(mu^{-1/m} + u\right)\right\} du \\
=\ & \frac{2^{(m+1)/2-2}}{\pi\sqrt{m}} \left(1 - C_{8m} x^{-2/(m+1)}\right)(J_{11} + J_{12})
\end{aligned}
$$

with a positive quantity C_{8m}, for all sufficiently large x. Using the estimates (3.27) and (3.28) we obtain

$$
\begin{aligned}
\overline{F}_{\Pi_{m+1}}(x) \geqslant\ & \frac{2^{(m+1)/2} - 1}{\sqrt{\pi(m+1)}}\, x^{-1/(m+1)} \exp\left\{-\frac{m+1}{2}\, x^{2/(m+1)}\right\} \\
& \times \left(1 - C_{9m} x^{-2/(m+1)}\right)
\end{aligned}
$$

with a positive quantity C_{9m} depending only on m, for large enough x, which, together with (3.29), implies the formula (3.13) in the case $n = m+1$. By the principle of induction, it follows that (3.13) holds for all $n \in \mathbb{N}$. Theorem 3.2 is proved. □

3.4 Examples

In this section, we consider two simulation-based examples involving products of normally distributed r.v.s. In both cases we compare the asymptotic estimates obtained from either Theorem 3.2 or Corollary 3.2 with the tail probabilities estimated by the MC simulations.

3.4.1 Example 1

Let us consider the product Π_4 of four i.i.d. r.v.s $\xi_1, \xi_2, \xi_3, \xi_4$ such that $\xi_i \overset{d}{=} \Phi$, $i = 1, \ldots, 4$. Due to Theorem 3.2 there exist two positive constants D_1 and D_2 such that for all $x \geqslant D_2$,

$$\frac{1}{\sqrt{\pi}} x^{-1/4} \mathrm{e}^{-2\sqrt{x}} \Big(1 - \frac{D_1}{\sqrt{x}}\Big) \leqslant \mathbb{P}(\Pi_4 > x) \leqslant \frac{1}{\sqrt{\pi}} x^{-1/4} \mathrm{e}^{-2\sqrt{x}} \Big(1 + \frac{D_1}{\sqrt{x}}\Big).$$

Simulation study revealed that it is possible to take $D_1 = 0.8$ and $D_2 = 6$ in this case. The simulated values of $\mathbb{P}(\Pi_4 > x)$, together with the upper and lower bounds (represented by blue lines) are presented in Figure 3.1.

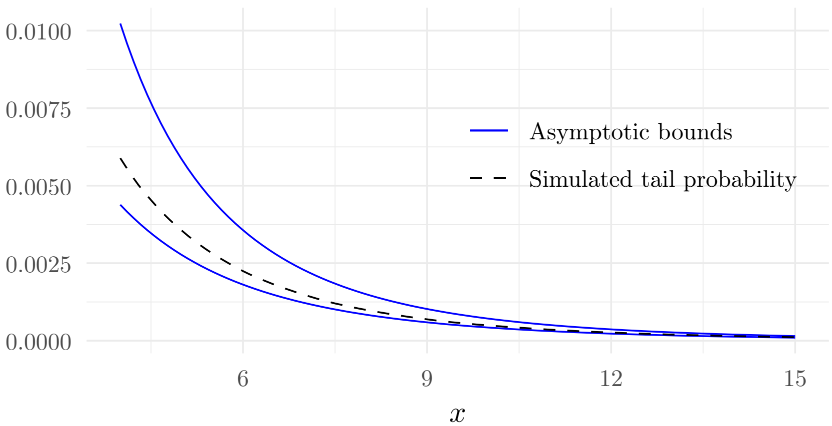

Figure 3.1: Simulated values and asymptotic bounds with $D_1 = 0.8$ for the tail probability of Π_4.

3.4.2 Example 2

This time we consider the product $\widehat{\Pi}_3$ of three i.i.d. zero-mean r.v.s ξ_1, ξ_2, ξ_3 such that $\xi_1 \overset{d}{=} \Phi_{\{0,1\}}$, $\xi_2 \overset{d}{=} \Phi_{\{0,2\}}$ and $\xi_3 \overset{d}{=} \Phi_{\{0,3\}}$. According to Corollary 3.2, there exist two positive constants $\widehat{D}_1$ and $\widehat{D}_2$ such that for $x \geqslant \widehat{D}_2$,

$$\mathbb{P}\left(\widehat{\Pi}_3 > x\right) \geqslant \left(1 - \frac{\widehat{D}_1}{\sqrt[3]{x^2}}\right) \sqrt{\frac{2}{3\pi}} \left(\frac{x}{6}\right)^{-1/3} \exp\left\{-\frac{3}{2}\left(\frac{x}{6}\right)^{2/3}\right\},$$

$$\mathbb{P}\left(\widehat{\Pi}_3 > x\right) \leqslant \left(1 + \frac{\widehat{D}_1}{\sqrt[3]{x^2}}\right)\sqrt{\frac{2}{3\pi}}\left(\frac{x}{6}\right)^{-1/3}\exp\left\{-\frac{3}{2}\left(\frac{x}{6}\right)^{2/3}\right\}.$$

Based on the simulations, the selected constants are $\widehat{D}_1 = 2$ and $\widehat{D}_2 = 15$. The simulated values of $\mathbb{P}(\widehat{\Pi}_3 > x)$, together with the upper and lower bounds (represented by blue lines), are depicted in Figure 3.2.

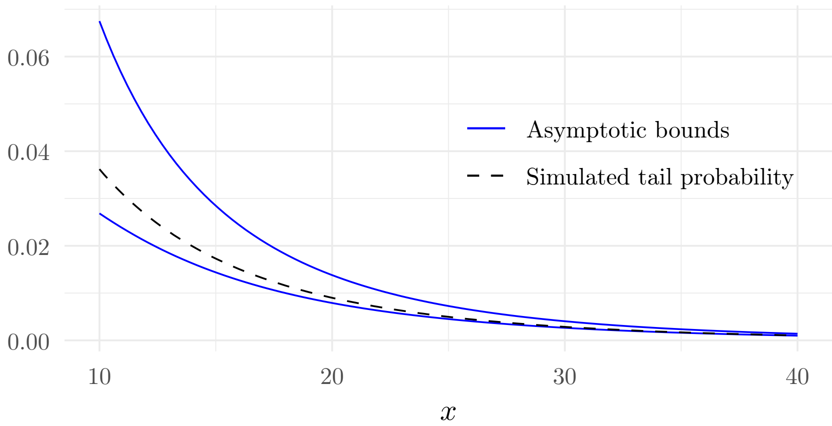

Figure 3.2: Simulated values and asymptotic bounds with $D_1 = 2$ for the tail probability of $\widehat{\Pi}_3$.

4 Asymptotic properties of sums with dominatedly varying increments

4.1 Related results

In this section, we review some of the results from the literature on the asymptotic properties of the tail probability and tail expectation of the random sums S_n^{ξ} and $S_n^{\theta\xi}$.

4.1.1 Asymptotics of tail probability

There are many papers in which the random summands $\xi_1, \ldots, \xi_n$ are assumed to be independent or identically distributed; see, for instance, the works of Tang and Tsitshiashivili [106], Goovaertz et al. [59], Wang et al. [113], Wang and Tang [114] and the references therein. In this subsection, however, we concentrate on the results in which such restrictive assumptions have been weakened.

To begin with, we list several results in which the exact asymptotic equivalence

$$\mathbb{P}\left(S_n^{\xi} > x\right) \underset{x\to\infty}{\sim} \sum_{k=1}^{n} \overline{F}_{\xi_k}(x) \tag{4.1}$$

was obtained. Geluk and Tang [56] achieved (4.1) for distributions $F_{\xi_k} \in \mathscr{D} \cap \mathscr{S}$ (see [56, Theorem 3.1]) under the pSQAI dependence structure (see Definition 2.9). Nearly at the same time, Chen and Yuen [25] obtained (4.1) (see [25, Theorem 3.1]) in the smaller class $\mathscr{C}$, but the pSQAI condition was replaced by the more general pQAI structure (see Definition 2.13). Moreover, the authors extended the result to the case of randomly weighted sums (see [25, Theorem 3.2]), resulting in the relation

$$\mathbb{P}\left(S_n^{\theta\xi} > x\right) \underset{x\to\infty}{\sim} \sum_{k=1}^{n} \overline{F}_{\theta_k\xi_k}(x)$$

under the following moment condition on random weights:

$$\max\{\mathbb{E}\theta_1^p \dots, \mathbb{E}\theta_n^p\} < \infty \quad \text{for some} \quad p > \max\{J^+_{F_{\xi_1}}, \dots, J^+_{F_{\xi_n}}\}. \quad (4.2)$$

Later, inspired by the aforementioned results of Chen and Yuen, Yi et al. [132] considered the tail probability asymptotics of the randomly weighted sum $S_n^{\theta\xi}$ when the r.v.s $\xi_1, \dots, \xi_n$ are dominatedly varying and follow the same pQAI dependence structure (see [132, Theorems 1 and 2]). It was shown that under (4.2) and the additional tail assumption (2.17) the following asymptotic bounds hold:

$$L_n^{\xi} \sum_{k=1}^{n} \overline{F}_{\theta_k\xi_k}(x) \underset{x\to\infty}{\lesssim} \mathbb{P}\left(S_n^{\theta\xi} > x\right) \underset{x\to\infty}{\lesssim} \frac{1}{L_n^{\xi}} \sum_{k=1}^{n} \overline{F}_{\theta_k\xi_k}(x), \quad (4.3)$$

where $L_n^{\xi} := \min\{L_{F_{\xi_1}}, \dots, L_{F_{\xi_n}}\}$. Cheng [26] managed to further tighten the bounds in (4.3) (see [26, Theorems 1.1 and 1.2]) by putting the L-indices of individual summands inside the bounding sums and thus obtaining

$$\sum_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\theta_k\xi_k}(x) \underset{x\to\infty}{\lesssim} \mathbb{P}\left(S_n^{\theta\xi} > x\right) \underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \overline{F}_{\theta_k\xi_k}(x), \quad (4.4)$$

where $F_{\xi_k} \in \mathscr{D}$ for all $k \in \{1, \dots, n\}$. The assumption (4.2) was replaced by a weaker condition (see [26, Assumption C and Remark 1.1]):

$$\lim_{x\to\infty} \frac{\overline{F}_{\theta_k}(yx)}{\overline{F}_{\theta_k\xi_k}(x)} = 0 \quad \text{for all} \quad k \in \{1, \dots, n\} \quad \text{and any} \quad y > 0. \quad (4.5)$$

Moreover, instead of the pQAI structure, two other dependence structures, namely, pTQAI (see Definition 2.10 and [26, Assumption B]) and pAI (see Definition 2.11) together with the condition (2.17) ([26, Assumption A]) were considered.

More recently, Jaunė et al. [62] reconsidered the asymptotic behaviour of the tail probability $\mathbb{P}(S_n^{\theta\xi} > x)$ with the pQAI r.v.s $\xi_1, \dots, \xi_n$

from the class $\mathscr{D}$. The result (see [62, Lemma 1]) mainly extended the results of Yi et al. [132, Theorems 1, 2], resulting in (4.3) under the moment condition (4.2), but without the additional assumption (2.17).

4.1.2 Asymptotics of tail expectations

Having reviewed the main results concerning the asymptotic behaviour of the tail probabilities $\mathbb{P}(S_n^{\xi} > x)$ and $\mathbb{P}(S_n^{\theta\xi} > x)$ we now turn to the asymptotics of the tail expectation of the random sums S_n^{ξ} and $S_n^{\theta\xi}$.

To begin with, Tang and Yuan [110] obtained the relation

$$\mathbb{E}\left(\theta_1\xi_1\mathbb{I}_{\{S_n^{\theta\xi}>x\}}\right) \underset{x\to\infty}{\sim} \mathbb{E}\left(\theta_1\xi_1\mathbb{I}_{\{\theta_1\xi_1>x\}}\right)$$

for i.i.d. r.v.s $\xi_1,\ldots,\xi_n$ from the class $\mathscr{D}\cap\mathscr{L}$ and random weights $\theta_1,\ldots,\theta_n$, satisfying $\mathbb{E}\theta_k^{p_k} < \infty$, $p_k > J^+_{F_{\xi_k}}$, $\overline{F}_{\theta_k\xi_k}(x) = O(\overline{F}_{\theta_1\xi_1}(x))$ for all $k\in\{1,\ldots,n\}$ (see [110, Theorem 4]). It was noted by Yang et al. [128] that under the additional condition $\overline{F}_{\theta_k\xi_k}(x) \asymp \overline{F}_{\theta_1\xi_1}(x)$ for all $k\in\{1,\ldots,n\}$, the relation

$$\mathbb{E}\left(S_n^{\theta\xi}\mathbb{I}_{\{S_n^{\theta\xi}>x\}}\right) \underset{x\to\infty}{\sim} \sum_{k=1}^{n}\mathbb{E}\left(\theta_k\xi_k\mathbb{I}_{\{\theta_k\xi_k>x\}}\right) \tag{4.6}$$

holds. Jaunė et al. in [62] later weakened the i.i.d. condition of the previous result allowing the pQAI or pSQAI dependence structures among the primary r.v.s $\xi_1,\ldots,\xi_n$, at the cost of exact asymptotics in (4.6). We state this result as a separate theorem.

Theorem 4.1 ([62, Theorem 2] and corrections in [76, Theorem 2, Remark 3])**.** *Let $\xi_1,\ldots,\xi_n$, $n\in\mathbb{N}$, be real-valued r.v.s such that $F_{\xi_k}\in\mathscr{D}$ for all $k\in\{1,\ldots,n\}$. If $\overline{F}_{\xi_k}(x)\asymp\overline{F}_{\xi_1}(x)$, $\mathbb{E}|\xi_1|<\infty$, $F_{\xi_k}(-x) = o(\overline{F}_{\xi_1}(x))$ for all $k\in\{1,\ldots,n\}$, and*

- $\xi_1,\ldots,\xi_n$ *are* pQAI*, then*

$$\mathbb{E}\left(S_n^{\xi}\mathbb{I}_{\{S_n^{\xi}>x\}}\right) \underset{x\to\infty}{\lesssim} \frac{1}{L_n^{\xi}}\sum_{k=1}^{n}\mathbb{E}\left(\xi_k\mathbb{I}_{\{\xi_k>x\}}\right);$$

- $\xi_1,\ldots,\xi_n$ *are* pSQAI*, then*

$$L_n^{\xi}\sum_{k=1}^{n}\mathbb{E}\left(\xi_k\mathbb{I}_{\{\xi_k>x\}}\right) \underset{x\to\infty}{\lesssim} \mathbb{E}\left(S_n^{\xi}\mathbb{I}_{\{S_n^{\xi}>x\}}\right) \underset{x\to\infty}{\lesssim} \frac{1}{L_n^{\xi}}\sum_{k=1}^{n}\mathbb{E}\left(\xi_k\mathbb{I}_{\{\xi_k>x\}}\right).$$

Now we turn to the recent results by Leipus et al. [76], which inspired our investigation. Note that this time, instead of the pQAI or pSQAI conditions, a new dependence structure Assumption B (see Definition 2.12) was considered. The results extended Theorem 4.1 to the cases of higher-order moments and corrected some mistakes left in [62] as well.

Theorem 4.2 (See [76, Theorem 4]). *Let $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, be real-valued r.v.s satisfying* Assumption B, *such that $F_{\xi_1} \in \mathscr{D}$, $\mathbb{E}\,|\xi_1|^m < \infty$ for some $m \in \mathbb{N}$ and*

$$\overline{F}_{\xi_k}(x) \asymp \overline{F}_{\xi_1}(x), \qquad \overline{F}_{\xi_k^-}(x) = O\left(\overline{F}_{\xi_1}(x)\right) \quad \textit{for all } k = 2, \ldots, n.$$

Then

$$\begin{aligned} L_n^{\xi} \sum_{k=1}^{n} \mathbb{E}\left(\xi_k^m \mathbb{I}_{\{\xi_k > x\}}\right) &\underset{x\to\infty}{\lesssim} \mathbb{E}\left((S_n^{\xi})^m \mathbb{I}_{\{S_n^{\xi} > x\}}\right) \\ &\underset{x\to\infty}{\lesssim} \frac{1}{L_n^{\xi}} \sum_{k=1}^{n} \mathbb{E}\left(\xi_k^m \mathbb{I}_{\{\xi_k > x\}}\right). \end{aligned}$$

Moreover, the results were extended to the general case of randomly weighted sums. However, this time, a quite restrictive assumption about random weights was made, namely that the random weights $\theta_1, \ldots, \theta_n$ are bounded.

Theorem 4.3 (See [76, Theorem 5]). *Let $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, be real-valued r.v.s satisfying* Assumption B, *such that $F_{\xi_1} \in \mathscr{D}$, $\mathbb{E}\,|\xi_1|^m < \infty$ for some $m \in \mathbb{N}$. Let $\theta_1, \ldots, \theta_n$ be nonnegative, nondegenerate at zero, bounded r.v.s independent of $\xi_1, \ldots, \xi_n$. If*

$$\overline{F}_{\theta_k\xi_k}(x) \asymp \overline{F}_{\theta_1\xi_1}(x), \quad \overline{F}_{\theta_k\xi_k^-}(x) = O\left(\overline{F}_{\theta_1\xi_1}(x)\right) \quad \textit{for all } k = 2, \ldots, n,$$

then

$$\begin{aligned} L_n^{\xi} \sum_{k=1}^{n} \mathbb{E}\left((\theta_k\xi_k)^m \mathbb{I}_{\{\theta_k\xi_k > x\}}\right) &\underset{x\to\infty}{\lesssim} \mathbb{E}\left((S_n^{\theta\xi})^m \mathbb{I}_{\{S_n^{\theta\xi} > x\}}\right) \\ &\underset{x\to\infty}{\lesssim} \frac{1}{L_n^{\xi}} \sum_{k=1}^{n} \mathbb{E}\left((\theta_k\xi_k)^m \mathbb{I}_{\{\theta_k\xi_k > x\}}\right). \end{aligned}$$

Finally, Sprindys and Šiaulys in [105] obtained exact asymptotic relations for the generalised tail moments of random sums S_n^ξ with pQAI consistently varying summands under the power function transformation. Moreover, the authors showed that the asymptotic behaviour of the tail moments of S_n^ξ depends on consistently varying distributed summands, but not on asymptotically lighter increments.

Theorem 4.4 (See [105, Theorem 4]). *Let $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, be real-valued* pQAI *r.v.s, such that $F_{\xi_L} \in \mathscr{C}$ for some $L \in \{1, \ldots, n\}$, and for the other indices $k \neq L$ let $F_{\xi_k} \in \mathscr{C}$ or $\mathbb{P}(|\xi_k| > x) = o\big(\overline{F}_{\xi_L}(x)\big)$. If $\max\limits_{1 \leqslant k \leqslant n} \mathbb{E}\big(\xi_k^+\big)^\alpha < \infty$ for some $\alpha \geqslant 0$, then for each $\beta \in [0, \alpha]$, we have*

$$\mathbb{E}\Big(\big(S_n^\xi\big)^\beta \mathbb{I}_{\{S_n^\xi > x\}}\Big) \underset{x\to\infty}{\sim} \sum_{k \in \mathcal{I}_n} \mathbb{E}\Big(\xi_k^\beta\, \mathbb{I}_{\{\xi_k > x\}}\Big), \tag{4.7}$$

and for each $\beta \in (0, \alpha]$, it holds that

$$\mathbb{E}\left(\Big(\big(S_n^\xi - x\big)^+\Big)^\beta\right) \underset{x\to\infty}{\sim} \sum_{k \in \mathcal{I}_n} \mathbb{E}\Big(\big(\xi_k - x\big)^+\Big)^\beta, \tag{4.8}$$

where $\mathcal{I}_n \subseteq \{1, \ldots, n\}$ is a subset of indices such that $F_{\xi_k} \in \mathscr{C}$ for all $k \in \mathcal{I}_n$.

4.2 Main results

In this section, we provide the main results regarding the asymptotic behaviour of the generalised tail moments of random sums with dominatedly varying summands. The first theorem below gives the asymptotic bounds of the tail moment of S_n^ξ under a transformation from a specific class of functions.

Theorem 4.5 (See [39, Theorem 2]). *Let $\xi_1, \ldots, \xi_n$ be* pQAI *real-valued r.v.s such that $F_{\xi_k} \in \mathscr{D}$ for all $k \in \{1, \ldots, n\}$ and let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a function that satisfies the following properties:*

(i) *φ is eventually[1] nondecreasing;*

(ii) *φ is eventually sub-homogeneous, i.e., $\varphi(2x) \leqslant c\varphi(x)$ for some positive constant c and all sufficiently large x;*

(iii) *φ is eventually differentiable;*

[1] We say that a property A holds *eventually* if it holds for all sufficiently large x.

(iv) *for each $k \in \{1,2,\ldots,n\}$, the generalised moment $\mathbb{E}(\varphi(\xi_k)\mathbb{I}_{\{\xi_k>x\}})$ is eventually finite.*

Then

$$\begin{aligned}\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\left(\varphi(\xi_k)\mathbb{I}_{\{\xi_k>x\}}\right) &\underset{x\to\infty}{\lesssim} \mathbb{E}\left(\varphi(S_n^{\xi})\mathbb{I}_{\{S_n^{\xi}>x\}}\right)\\ &\underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}}\mathbb{E}\left(\varphi(\xi_k)\mathbb{I}_{\{\xi_k>x\}}\right).\end{aligned} \tag{4.9}$$

Remark 4.1. *It should be noted that requirements (i), (ii), (iii) impose conditions solely on the function φ, whereas condition (iv) involves not only the function φ but depends on the collection of r.v.s $\{\xi_1,\ldots,\xi_n\}$ as well.*

The following theorem is a generalisation of Theorem 4.5 to the case of randomly weighted sums $S_n^{\theta\xi}$.

Theorem 4.6 (See [39, Theorem 3])**.** *Let $\xi_1,\ldots,\xi_n$, $n\in\mathbb{N}$, be* pQAI *real-valued r.v.s such that $F_{\xi_k}\in\mathscr{D}$ for all $k\in\{1,\ldots,n\}$ and let $\theta_1,\ldots,\theta_n$ be arbitrarily dependent, nonnegative, nondegenerate at zero r.v.s such that*

$$\max\{\mathbb{E}\theta_1^p,\ldots,\mathbb{E}\theta_n^p\}<\infty \quad \text{for some} \quad p>\max\{J^+_{F_{\xi_1}},\ldots,J^+_{F_{\xi_n}}\}. \tag{4.10}$$

Additionally, let the collections $\{\xi_1,\ldots,\xi_n\}$ and $\{\theta_1,\ldots,\theta_n\}$ be independent and $\varphi:\mathbb{R}\to\mathbb{R}_+$ be a function that satisfies properties (i), (ii), (iii) *from Theorem 4.5 as well as the following one:*

(iv)* *the generalised moment $\mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k>x\}}\right)$ is eventually finite for all $k\in\{1,\ldots,n\}$.*

Then

$$\begin{aligned}\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k>x\}}\right) &\underset{x\to\infty}{\lesssim} \mathbb{E}\left(\varphi(S_n^{\theta\xi})\mathbb{I}_{\{S_n^{\theta\xi}>x\}}\right)\\ &\underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}}\mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k>x\}}\right).\end{aligned} \tag{4.11}$$

We observe that the case where φ is a power function transformation is a special case of Theorems 4.5 and 4.6.

Corollary 4.1 (See [40, Theorem 4])**.** *Let $\xi_1,\ldots,\xi_n$, $n\in\mathbb{N}$, be* pQAI *real-valued r.v.s such that $F_{\xi_k}\in\mathscr{D}$ for all $k\in\{1,2,\ldots,n\}$ and let*

$\theta_1, \ldots, \theta_n$ be arbitrarily dependent, nonnegative, nondegenerate at zero r.v.s satisfying (4.10)*. If the collections $\{\xi_1, \ldots, \xi_n\}$ and $\{\theta_1, \ldots, \theta_n\}$ are independent and $\mathbb{E}(\theta_k|\xi_k|)^\alpha < \infty$ for all $k \in \{1,2,\ldots,n\}$ and some $\alpha \in [0, \infty)$, then*

$$\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\left((\theta_k\xi_k)^\alpha \mathbb{I}_{\{\theta_k\xi_k > x\}}\right) \underset{x\to\infty}{\lesssim} \mathbb{E}\left((S_n^{\theta\xi})^\alpha \mathbb{I}_{\{S_n^{\theta\xi} > x\}}\right)$$
$$\underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\left((\theta_k\xi_k)^\alpha \mathbb{I}_{\{\theta_k\xi_k > x\}}\right).$$

We note that our main findings improve the related results reviewed in Section 4.1 in many ways. Firstly, we put the L-indices of individual summands inside the bounding sums in the equations (4.9) and (4.11) resulting in tighter asymptotic bounds. In addition, Assumption B used in Theorems 4.2 and 4.3 is replaced by a less restrictive pQAI dependence structure (see also Proposition 2.1) and random weights $\theta_1, \ldots, \theta_n$ need not be bounded as considered in [76]. Finally, we extend our analysis beyond power functions with nonnegative integer exponents (as in [76]) to a broader class of transformations.

We conclude the section with several remarks.

Remark 4.2. *By narrowing the class $\mathscr{D}$ to the class $\mathscr{C}$, we obtain an exact asymptotic equivalence in* (4.11)*. That is, if $F_{\xi_k} \in \mathscr{C}$ for all $k \in \{1, \ldots, n\}$ and all other conditions of Theorem 4.6 hold, then*

$$\sum_{k=1}^{n} \mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k > x\}}\right) \underset{x\to\infty}{\sim} \mathbb{E}\left(\varphi(S_n^{\theta\xi})\mathbb{I}_{\{S_n^{\theta\xi} > x\}}\right).$$

The above relation follows directly from the definitions of L-index and class $\mathscr{C}$.

Remark 4.3. *When $\alpha = 0$ in Corollary 4.1 or $\varphi \equiv 1$ in Theorem 4.6, we obtain the following asymptotic bounds for the tail probabilities:*

$$\begin{aligned} \sum_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\theta_k\xi_k}(x) &\leqslant \sum_{k=1}^{n} L_{F_{\theta_k\xi_k}} \overline{F}_{\theta_k\xi_k}(x) \underset{x\to\infty}{\lesssim} \mathbb{P}\left(S_n^{\theta\xi} > x\right), \\ \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \overline{F}_{\theta_k\xi_k}(x) &\geqslant \sum_{k=1}^{n} \frac{1}{L_{F_{\theta_k\xi_k}}} \overline{F}_{\theta_k\xi_k}(x) \underset{x\to\infty}{\gtrsim} \mathbb{P}\left(S_n^{\theta\xi} > x\right). \end{aligned} \tag{4.12}$$

Remark 4.4. *Under the same conditions as in Theorem 4.6 we can obtain the asymptotic bounds for the conditional expectation* $\mathbb{E}(\varphi(S_n^{\theta\xi}) \mid S_n^{\theta\xi} > x)$. *Combining* (4.11) *with* (4.12) *we get*

$$\begin{aligned} \frac{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{E}(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k>x\}})}{\sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}} \overline{F}_{\theta_k\xi_k}(x)} &\underset{x\to\infty}{\lesssim} \mathbb{E}\left(\varphi(S_n^{\theta\xi}) \mid S_n^{\theta\xi} > x\right), \\ \frac{\sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}} \mathbb{E}(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k>x\}})}{\sum_{k=1}^n L_{F_{\xi_k}} \overline{F}_{\theta_k\xi_k}(x)} &\underset{x\to\infty}{\gtrsim} \mathbb{E}\left(\varphi(S_n^{\theta\xi}) \mid S_n^{\theta\xi} > x\right). \end{aligned} \tag{4.13}$$

By the min-max inequality (2.43), *we can express* (4.13) *fully in conditional expectations at the cost of the tightness of initial bounds:*

$$\begin{aligned} \min_{1\leqslant k\leqslant n} \left\{ L_{F_{\xi_k}}^2 \mathbb{E}\left(\varphi(\theta_k\xi_k) \mid \theta_k\xi_k > x\right)\right\} &\underset{x\to\infty}{\lesssim} \mathbb{E}\left(\varphi(S_n^{\theta\xi}) \mid S_n^{\theta\xi} > x\right), \\ \max_{1\leqslant k\leqslant n} \left\{ \frac{1}{L_{F_{\xi_k}}^2} \mathbb{E}\left(\varphi(\theta_k\xi_k) \mid \theta_k\xi_k > x\right)\right\} &\underset{x\to\infty}{\gtrsim} \mathbb{E}\left(\varphi(S_n^{\theta\xi}) \mid S_n^{\theta\xi} > x\right). \end{aligned}$$

4.3 Proofs of the main results

To prove Theorems 4.5 and 4.6 we need several auxiliary assertions.

Lemma 4.1. *Let* ξ *be a real-valued r.v. with d.f. supported on* $\mathbb{R}$ *and let* $\varphi\colon [a,\infty) \to \mathbb{R}$ *be a differentiable function such that*

$$\int_{[a,\infty)} \left(\int_a^z |\varphi'(t)| \mathrm{d}t\right) \mathrm{d}F_\xi(z) < \infty \tag{4.14}$$

for some positive a. *Then*

$$\mathbb{E}\left(\varphi(\xi)\mathbb{I}_{\{\xi>x\}}\right) = \varphi(x)\overline{F}_\xi(x) + \int_x^\infty \varphi'(t)\overline{F}_\xi(t)\mathrm{d}t \tag{4.15}$$

for an arbitrary $x \geqslant a$.

Proof. The assertion of the lemma can be derived from Lemma 2.5 or can be proved directly using Fubini's theorem. We choose the latter path. For a given $x \geqslant a$, we have

$$\begin{aligned} \mathbb{E}\left((\varphi(\xi)-\varphi(x))\mathbb{I}_{\{\xi>x\}}\right) &= \int_{\mathbb{R}} (\varphi(u)-\varphi(x))\mathbb{I}_{\{u>x\}} \mathrm{d}F_\xi(u) \\ &= \int_{(x,\infty)} (\varphi(u)-\varphi(x)) \mathrm{d}F_\xi(u) \end{aligned}$$

$$= \int_{(x,\infty)} \Big(\int_x^u \varphi'(t)\mathrm{d}t \Big) \mathrm{d}F_\xi(u).$$

Condition (4.14) enables us to use the Fubini-Tonelli theorem. According to this theorem, we obtain

$$\begin{aligned} \int_{(x,\infty)} \Big(\int_x^u \varphi'(t)\mathrm{d}t \Big) \mathrm{d}F_\xi(u) &= \int_x^\infty \Big(\int_{(t,\infty)} \varphi'(t)\mathrm{d}F_\xi(u) \Big) \mathrm{d}t \\ &= \int_x^\infty \varphi'(t) \Big(\int_{(t,\infty)} \mathrm{d}F_\xi(u) \Big) \mathrm{d}t \\ &= \int_x^\infty \varphi'(t)(1 - F_\xi(t))\mathrm{d}t \\ &= \int_x^\infty \varphi'(t)\overline{F}_\xi(t)\mathrm{d}t. \end{aligned}$$

The derived two formulas imply the equality in (4.15). This ends the proof of Lemma 4.1. □

The following assertion, which comes as a special case of Theorem 4.5 by taking $\varphi(x) \equiv 1$, is crucial for the proofs of the theorems in the general case.

Lemma 4.2 (See [40, Lemma 2]). *Let $\xi_1, \ldots, \xi_n$ be* pQAI *real-valued r.v.s, such that $F_{\xi_k} \in \mathscr{D}$ for all $k \in \{1, \ldots, n\}$. Then*

$$\sum_{k=1}^n L_{F_{\xi_k}} \overline{F}_{\xi_k}(x) \underset{x\to\infty}{\lesssim} \mathbb{P}(S_n^\xi > x) \underset{x\to\infty}{\lesssim} \sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}} \overline{F}_{\xi_k}(x). \tag{4.16}$$

Proof. The case $n = 1$ in (4.16) follows directly from the definition of $L_{F_{\xi_k}}$. Let $n \geqslant 2$. First, consider the upper asymptotic bound in (4.16).

For an arbitrary $\delta \in (0, 1)$,

$$\begin{aligned} \mathbb{P}(S_n^\xi > x) &= \mathbb{P}\left(S_n^\xi > x, \bigcup_{k=1}^n \{\xi_k > (1-\delta)x\} \cup \bigcap_{k=1}^n \{\xi_k \leqslant (1-\delta)x\} \right) \\ &\leqslant \mathbb{P}\left(\bigcup_{k=1}^n \{\xi_k > (1-\delta)x\} \right) \\ &\qquad + \mathbb{P}\left(S_n^\xi > x, \bigcap_{k=1}^n \{\xi_k \leqslant (1-\delta)x\} \right) \\ &\leqslant \sum_{k=1}^n \overline{F}_{\xi_k}((1-\delta)x) + \mathbb{P}\left(S_n^\xi > x, \bigcap_{k=1}^n \{\xi_k \leqslant (1-\delta)x\} \right) \end{aligned}$$

$$=: \sum_{k=1}^{n} \overline{F}_{\xi_k}((1-\delta)x) + \mathcal{A}(x,\delta).$$

By observing that for all $k \in \{1,\ldots,n\}$,

$$\left\{S_n^{\xi} > x, \xi_k \leqslant (1-\delta)x\right\} \subseteq \left\{S_n^{\xi} - \xi_k > \delta x\right\},$$

we can estimate the term $\mathcal{A}(x,\delta)$ as follows:

$$\begin{aligned}
\mathcal{A}(x,\delta) &= \mathbb{P}\left(S_n^{\xi} > x, \bigcup_{k=1}^{n}\left\{\xi_k > \frac{x}{n}\right\}, \bigcap_{k=1}^{n}\{\xi_k \leqslant (1-\delta)x\}\right)\\
&\leqslant \sum_{k=1}^{n} \mathbb{P}\left(\xi_k > \frac{x}{n}, S_n^{\xi} - \xi_k > \delta x\right)\\
&= \sum_{k=1}^{n} \mathbb{P}\left(\xi_k > \frac{x}{n}, S_n^{\xi} - \xi_k > \delta x, \bigcup_{l=1,l\neq k}^{n}\left\{\xi_l > \frac{\delta x}{n-1}\right\}\right)\\
&\leqslant \sum_{k=1}^{n} \mathbb{P}\left(\xi_k > \frac{x}{n}, \bigcup_{l=1,l\neq k}^{n}\left\{\xi_l > \frac{\delta x}{n-1}\right\}\right)\\
&\leqslant \sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n} \mathbb{P}\left(\xi_k > \frac{x}{n}, \xi_l > \frac{\delta x}{n-1}\right)\\
&\leqslant \sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n} \mathbb{P}\left(\xi_k > \delta_1 x, \xi_l > \delta_1 x\right),
\end{aligned}$$

where the last inequality holds for all $\delta_1 \in \left(0, \min\left\{\frac{1}{n}, \frac{\delta}{n-1}\right\}\right)$.

Consequently,

$$\begin{aligned}
\frac{\mathbb{P}(S_n^{\xi} > x)}{\sum_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\overline{F}_{\xi_k}(x)} &\leqslant \frac{\sum_{k=1}^{n}\overline{F}_{\xi_k}((1-\delta)x)}{\sum_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\overline{F}_{\xi_k}(x)}\\
&\qquad + \frac{\sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n}\mathbb{P}\left(\xi_k > \delta_1 x, \xi_l > \delta_1 x\right)}{\sum_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\overline{F}_{\xi_k}(x)}\\
&=: \mathcal{I}_1(x,\delta) + \mathcal{I}_2(x,\delta_1).
\end{aligned}$$

Lemma 2.4 implies

$$\mathcal{I}_1(x,\delta) \leqslant \max_{1\leqslant k\leqslant n}\left\{L_{F_{\xi_k}}\frac{\overline{F}_{\xi_k}((1-\delta)x)}{\overline{F}_{\xi_k}(x)}\right\}.$$

Observing that

$$\sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n}\left(\overline{F}_{\xi_k}(\delta_1 x)+\overline{F}_{\xi_l}(\delta_1 x)\right)\leqslant 2(n-1)\sum_{k=1}^{n}\overline{F}_{\xi_k}(\delta_1 x)$$

and taking into account Lemma 2.4, we similarly obtain

$$\begin{aligned}\mathcal{I}_2(x,\delta_1)&=\frac{\sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n}\mathbb{P}\left(\xi_k>\delta_1 x,\xi_l>\delta_1 x\right)}{\sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n}\left(\overline{F}_{\xi_k}(\delta_1 x)+\overline{F}_{\xi_l}(\delta_1 x)\right)}\\&\qquad\times\frac{\sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n}\left(\overline{F}_{\xi_k}(\delta_1 x)+\overline{F}_{\xi_l}(\delta_1 x)\right)}{\sum_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\overline{F}_{\xi_k}(x)}\\&\leqslant\max_{1\leqslant k\neq l\leqslant n}\left\{\frac{\mathbb{P}\left(\xi_k>\delta_1 x,\xi_l>\delta_1 x\right)}{\overline{F}_{\xi_k}(\delta_1 x)+\overline{F}_{\xi_l}(\delta_1 x)}\right\}\\&\qquad\times 2\,(n-1)\max_{1\leqslant k\leqslant n}\left\{L_{F_{\xi_k}}\frac{\overline{F}_{\xi_k}(\delta_1 x)}{\overline{F}_{\xi_k}(x)}\right\}.\end{aligned}\tag{4.17}$$

The fact that $F_{\xi_k}\in\mathscr{D}$ for all $k\in\{1,\ldots,n\}$ and the pQAI condition for the r.v.s $\xi_1,\ldots,\xi_n$ imply:

$$\limsup_{x\to\infty}\mathcal{I}_1(x,\delta)\leqslant\max_{1\leqslant k\leqslant n}\left\{L_{F_{\xi_k}}\lim_{\delta\downarrow 0}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}((1-\delta)x)}{\overline{F}_{\xi_k}(x)}\right\}=1,\tag{4.18}$$

$$\begin{aligned}\limsup_{x\to\infty}\mathcal{I}_2(x,\delta_1)&\leqslant 2\,(n-1)\max_{1\leqslant k\neq l\leqslant n}\left\{\limsup_{x\to\infty}\frac{\mathbb{P}\left(\xi_k>\delta_1 x,\xi_l>\delta_1 x\right)}{\overline{F}_{\xi_k}(\delta_1 x)+\overline{F}_{\xi_l}(\delta_1 x)}\right\}\\&\qquad\times\max_{1\leqslant k\leqslant n}\left\{L_{F_{\xi_k}}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(\delta_1 x)}{\overline{F}_{\xi_k}(x)}\right\}=0.\end{aligned}\tag{4.19}$$

Thus, by letting $\delta\downarrow 0$, from the estimates (4.18), (4.19) and the definition of indices $L_{F_{\xi_k}}$ we get the upper bound in (4.16):

$$\limsup_{x\to\infty}\frac{\mathbb{P}(S_n^{\xi}>x)}{\sum_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\overline{F}_{\xi_k}(x)}\leqslant 1.\tag{4.20}$$

Consider now the lower asymptotic bound in (4.16). Again, choose an arbitrary $\delta \in (0,1)$. By the Bonferroni inequality, for this δ, we get

$$\begin{aligned}
\mathbb{P}\left(S_n^\xi > x\right) &\geqslant \mathbb{P}\left(S_n^\xi > x, \bigcup_{k=1}^{n} \{\xi_k > (1+\delta)x\}\right) \\
&\geqslant \sum_{k=1}^{n} \mathbb{P}\left(S_n^\xi > x, \xi_k > (1+\delta)x\right) \\
&\quad - \sum_{k=1}^{n} \sum_{l=1, l\neq k}^{n} \mathbb{P}\left(\xi_k > (1+\delta)x, \xi_l > (1+\delta)x\right) \\
&=: \mathcal{A}_1(x,\delta) - \mathcal{A}_2(x,\delta). \qquad (4.21)
\end{aligned}$$

For the first summand in (4.21), observe that for all $k \in \{1,\ldots,n\}$,

$$\left\{S_n^\xi > x, \xi_k > (1+\delta)x\right\} \supseteq \left\{S_n^\xi - \xi_k > -\delta x, -\delta x > x - \xi_k\right\},$$

and thus we obtain:

$$\begin{aligned}
\mathcal{A}_1(x,\delta) &\geqslant \sum_{k=1}^{n} \mathbb{P}\left(S_n^\xi - \xi_k > -\delta x, \xi_k > (1+\delta)x\right) \\
&= \sum_{k=1}^{n} \overline{F}_{\xi_k}((1+\delta)x) - \sum_{k=1}^{n} \mathbb{P}\left(S_n^\xi - \xi_k \leqslant -\delta x, \xi_k > (1+\delta)x\right) \\
&=: \mathcal{A}_{11}(x,\delta) - \mathcal{A}_{12}(x,\delta). \qquad (4.22)
\end{aligned}$$

For the second term in (4.22), we get

$$\begin{aligned}
\mathcal{A}_{12}(x,\delta) &\leqslant \sum_{k=1}^{n} \mathbb{P}\left(\bigcup_{l=1, l\neq k}^{n} \left\{\xi_l \leqslant -\frac{\delta x}{n-1}\right\}, \xi_k > (1+\delta)x\right) \\
&\leqslant \sum_{k=1}^{n} \sum_{l=1, l\neq k}^{n} \mathbb{P}\left(\xi_k > (1+\delta)x, \xi_l^- \geqslant \frac{\delta x}{n-1}\right) \\
&\leqslant \sum_{k=1}^{n} \sum_{l=1, l\neq k}^{n} \mathbb{P}\left(\xi_k > \delta_2 x, \xi_l^- \geqslant \delta_2 x\right), \qquad (4.23)
\end{aligned}$$

where the inequality (4.23) holds for all $\delta_2 \in (0, \frac{\delta}{n-1})$. From (4.21), (4.22), (4.23) we obtain

$$\frac{\mathbb{P}(S_n^\delta > x)}{\sum_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\xi_k}(x)} \geqslant \frac{\mathcal{A}_{11}(x,\delta)}{\sum_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\xi_k}(x)}$$

$$
\begin{aligned}
&- \frac{\sum_{k=1}^{n} \sum_{l=1, l\neq k}^{n} \mathbb{P}\left(\xi_k > \delta_2 x, \xi_l^- \geqslant \delta_2 x\right)}{\sum_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\xi_k}(x)} \\
&- \frac{\mathcal{A}_2(x,\delta)}{\sum_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\xi_k}(x)} \\
&=: \mathcal{J}_1(x,\delta) - \mathcal{J}_2(x,\delta_2) - \mathcal{J}_3(x,\delta).
\end{aligned}
$$

We estimate each term $\mathcal{J}_i(x,\delta)$, $i = 1,2,3$, separately. For the case $i = 1$, using Lemma 2.4 we get

$$
\mathcal{J}_1(x,\delta) \geqslant \min_{1\leqslant k\leqslant n} \left\{ \frac{\overline{F}_{\xi_k}((1+\delta)x)}{L_{F_{\xi_k}} \overline{F}_{\xi_k}(x)} \right\}.
$$

For $\mathcal{J}_2(x,\delta_2)$, similarly to the derivation of (4.17), we obtain

$$
\begin{aligned}
\mathcal{J}_2(x,\delta_2) \leqslant 2\,(n-1) \max_{1\leqslant k\neq l\leqslant n} &\left\{ \frac{\mathbb{P}\left(\xi_k > \delta_2 x, \xi_l^- > \delta_2 x\right)}{\overline{F}_{\xi_k}(\delta_2 x) + \overline{F}_{\xi_l}(\delta_2 x)} \right\} \\
&\times \max_{1\leqslant k\leqslant n} \left\{ \frac{\overline{F}_{\xi_k}(\delta_2 x)}{L_{F_{\xi_k}} \overline{F}_{\xi_k}(x)} \right\}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathcal{J}_3(x,\delta) &\leqslant 2\,(n-1) \max_{1\leqslant k\neq l\leqslant n} \left\{ \frac{\mathbb{P}\left(\xi_k > (1+\delta)x, \xi_l > (1+\delta)x\right)}{\overline{F}_{\xi_k}((1+\delta)x) + \overline{F}_{\xi_l}((1+\delta)x)} \right\} \\
&\quad \times \max_{1\leqslant k\leqslant n} \left\{ \frac{\overline{F}_{\xi_k}((1+\delta)x)}{L_{F_{\xi_k}} \overline{F}_{\xi_k}(x)} \right\} \\
&\leqslant \max_{1\leqslant k\leqslant n} \left\{ \frac{2(n+1)}{L_{F_{\xi_k}}} \right\} \\
&\quad \times \max_{1\leqslant k\neq l\leqslant n} \left\{ \frac{\mathbb{P}\left(\xi_k > (1+\delta)x, \xi_l > (1+\delta)x\right)}{\overline{F}_{\xi_k}((1+\delta)x) + \overline{F}_{\xi_l}((1+\delta)x)} \right\}.
\end{aligned}
$$

From the fact that $F_{\xi_k} \in \mathscr{D}$ for all $k \in \{1,\ldots,n\}$ and the pQAI condition for the r.v.s $\{\xi_1,\ldots,\xi_n\}$ we get the following estimates:

$$
\liminf_{x\to\infty} \mathcal{J}_1(x,\delta) \geqslant \min_{1\leqslant k\leqslant n} \left\{ \frac{1}{L_{F_{\xi_k}}} \liminf_{x\to\infty} \frac{\overline{F}_{\xi_k}((1+\delta)x)}{\overline{F}_{\xi_k}(x)} \right\}, \tag{4.24}
$$

$$
\limsup_{x\to\infty} \mathcal{J}_2(x,\delta_2) \leqslant 2\,(n-1) \max_{1\leqslant k\neq l\leqslant n} \left\{ \limsup_{x\to\infty} \frac{\mathbb{P}\left(\xi_k > \delta_2 x, \xi_l^- > \delta_2 x\right)}{\overline{F}_{\xi_k}(\delta_2 x) + \overline{F}_{\xi_l}(\delta_2 x)} \right\}
$$

$$\times \max_{1\leqslant k\leqslant n}\left\{\frac{1}{L_{F_{\xi_k}}}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(\delta_2 x)}{\overline{F}_{\xi_k}(x)}\right\}, \tag{4.25}$$

$$\limsup_{x\to\infty}\mathcal{J}_3(x,\delta)\leqslant \max_{1\leqslant k\leqslant n}\left\{\frac{2(n+1)}{L_{F_{\xi_k}}}\right\}$$
$$\times \max_{1\leqslant k\neq l\leqslant n}\left\{\limsup_{x\to\infty}\frac{\mathbb{P}\left(\xi_k>(1+\delta)x,\xi_l>(1+\delta)x\right)}{\overline{F}_{\xi_k}((1+\delta)x)+\overline{F}_{\xi_l}((1+\delta)x)}\right\}. \tag{4.26}$$

Thus, letting $\delta\downarrow 0$, from the estimates (4.24), (4.25), (4.26) we obtain the lower asymptotic bound in (4.16):

$$\limsup_{x\to\infty}\frac{\mathbb{P}(S_n^\xi>x)}{\sum\limits_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\overline{F}_{\xi_k}(x)}\geqslant 1.$$

This finish the proof of Lemma 4.2 □

Proof of Theorem 4.5. The case $n=1$ follows directly from the definition of coefficient $L_{F_{\xi_1}}$. Let us suppose $n\geqslant 2$. Since n is fixed, the conditions of the theorem imply that requirements (i)–(iv) are satisfied for all $x\geqslant N>0$, when N is sufficiently large.

Due to properties (i) and (ii) we obtain $\varphi(nx)=O(\varphi(x))$. Indeed, since $\varphi(2x)\leqslant c(\varphi(x))$ for $x\geqslant N$, it follows that for any given $m\in\mathbb{N}$, the following inequalities hold for $x\geqslant 2^m N/n$:

$$\varphi(nx)\leqslant c\varphi\left(\frac{nx}{2}\right)\leqslant\ldots\leqslant c^{m-1}\varphi\left(\frac{nx}{2^{m-1}}\right)\leqslant c^m\varphi\left(\frac{nx}{2^m}\right).$$

By choosing m such that $\frac{n}{2^m}\leqslant 1$ and using the fact that φ is eventually nondecreasing, we get

$$\varphi(nx)\leqslant c^m\varphi\left(\frac{nx}{2^m}\right)\leqslant c^m\varphi(x)\leqslant c^{\lfloor\log_2 n\rfloor+1}\varphi(x)$$

for $x\geqslant K$ with some $K\geqslant N$. Therefore, property (iv) together with the derived estimate implies that

$$\mathbb{E}\Big(\varphi(S_n^\xi)\mathbb{I}_{\{S_n^\xi>\widehat{M}\}}\Big)<\infty$$

for some $\widehat{M}>0$.

Indeed, there exists $\widehat{K}\geqslant K$ such that $0<\varphi(x_1)\leqslant\varphi(x_2)$ for $\widehat{K}<$

$x_1 < x_2$ and

$$\mathbb{E}\Big(\varphi(\xi_k)\mathbb{I}_{\{\xi_k > \widehat{K}\}}\Big) < \infty$$

for $k \in \{1, \ldots, n\}$. Thus, for $\widehat{M} = n\widehat{K}$, we get

$$\begin{aligned}
\mathbb{E}\left(\varphi(S_n^\xi)\mathbb{I}_{\{S_n^\xi > \widehat{M}\}}\right) &\leqslant \mathbb{E}\left(\varphi(n\max\{\xi_1, \ldots, \xi_n\})\mathbb{I}_{\{\max\{\xi_1,\ldots,\xi_n\} > \widehat{K}\}}\right) \\
&\leqslant 2^{\log_2 n+1}\sum_{k=1}^{n}\mathbb{E}\left(\varphi(n\xi_k)\mathbb{I}_{\{\xi_k > \widehat{K}\}}\right) \\
&\leqslant \sum_{k=1}^{n}\mathbb{E}\left(\varphi(\xi_k)\mathbb{I}_{\{\xi_k > \widehat{K}\}}\right) < \infty.
\end{aligned}$$

Hence, due to properties (i) and (iii), there exists $M > \widehat{M}$ such that

$$\begin{aligned}
\int_{[M,\infty)}\left(\int_M^u |\varphi'(t)|\mathrm{d}t\right)\mathrm{d}F_{S_n^\xi}(u) &= \int_{[M,\infty)}\left(\int_M^u \varphi'(t)\mathrm{d}t\right)\mathrm{d}F_{S_n^\xi}(u) \\
&= \mathbb{E}\left(\varphi(S_n^\xi)\mathbb{I}_{\{S_n^\xi \geqslant M\}} - \varphi(M)\right) < \infty.
\end{aligned}$$

It follows that Lemma 4.1 is applicable for the truncated sum $S_n^\xi\mathbb{I}_{\{S_n^\xi > x\}}$. By Lemmas 2.4 and 4.1, we have for x large enough ($x \geqslant M$),

$$\begin{aligned}
\frac{\mathbb{E}\left(\varphi(S_n^\xi)\mathbb{I}_{\{S_n^\xi > x\}}\right)}{\sum\limits_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\mathbb{E}\left(\varphi(\xi_k)\mathbb{I}_{\{\xi_k > x\}}\right)} &= \frac{\varphi(x)\mathbb{P}\left(S_n^\xi > x\right) + \int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi > u\right)\mathrm{d}u}{\sum\limits_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\left(\varphi(x)\overline{F}_{\xi_k}(x) + \int\limits_x^\infty \varphi'(u)\overline{F}_{\xi_k}(u)\mathrm{d}u\right)} \\
&\leqslant \max\left\{\frac{\mathbb{P}\left(S_n^\xi > x\right)}{\sum\limits_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\overline{F}_{\xi_k}(x)}, \frac{\int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi > u\right)\mathrm{d}u}{\int\limits_x^\infty \varphi'(u)\sum\limits_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\overline{F}_{\xi_k}(u)\mathrm{d}u}\right\} \\
&=: \max\left\{\mathcal{C}_1(x), \mathcal{C}_2(x)\right\}. \qquad (4.27)
\end{aligned}$$

By Lemma 4.2, we obtain

$$\limsup_{x\to\infty}\mathcal{C}_1(x) \leqslant 1,$$

and for the term $\mathcal{C}_2(x)$, we have

$$\limsup_{x\to\infty}\mathcal{C}_2(x) = \limsup_{x\to\infty}\frac{\int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi > u\right)\mathrm{d}u}{\int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi > u\right)\frac{\sum_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\overline{F}_{\xi_k}(u)}{\mathbb{P}\left(S_n^\xi > u\right)}\mathrm{d}u}$$

$$\leqslant \limsup_{x\to\infty} \sup_{u\geqslant x} \frac{\mathbb{P}\left(S_n^\xi > u\right)}{\sum\limits_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \overline{F}_{\xi_k}(u)}$$

$$\leqslant \limsup_{x\to\infty} \frac{\mathbb{P}\left(S_n^\xi > x\right)}{\sum\limits_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \overline{F}_{\xi_k}(x)} \leqslant 1, \tag{4.28}$$

where the last inequality follows from Lemma 4.2 as well. The desired upper estimate in (4.9) now follows from (4.27). The asymptotic lower bound in (4.9) follows similarly. Indeed, in the same fashion we obtain

$$\frac{\mathbb{E}\left(\varphi(S_n^\xi)\mathbb{I}_{\{S_n^\xi > x\}}\right)}{\sum\limits_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\left(\varphi(\xi_k)\mathbb{I}_{\{\xi_k > x\}}\right)} = \frac{\varphi(x)\mathbb{P}\left(S_n^\xi > x\right) + \int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi > u\right)\mathrm{d}u}{\sum\limits_{k=1}^{n} L_{F_{\xi_k}} \left(\varphi(x)\overline{F}_{\xi_k}(x) + \int\limits_x^\infty \varphi'(u)\overline{F}_{\xi_k}(u)\mathrm{d}u\right)}$$

$$\geqslant \min\left\{\frac{\mathbb{P}\left(S_n^\xi > x\right)}{\sum\limits_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\xi_k}(x)}, \frac{\int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi > u\right)\mathrm{d}u}{\int\limits_x^\infty \varphi'(u) \sum\limits_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\xi_k}(u)\mathrm{d}u}\right\}$$

$$=: \min\left\{\mathcal{C}_3(x), \mathcal{C}_4(x)\right\}. \tag{4.29}$$

By Lemma 4.2, we obtain

$$\liminf_{x\to\infty} \mathcal{C}_3(x) \geqslant 1,$$

and for the term $\mathcal{C}_4(x)$, we get

$$\liminf_{x\to\infty} \mathcal{C}_4(x) = \liminf_{x\to\infty} \frac{\int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi > u\right)\mathrm{d}u}{\int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi > u\right) \frac{\sum_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\xi_k}(u)}{\mathbb{P}\left(S_n^\xi > u\right)}\mathrm{d}u}$$

$$\geqslant \liminf_{x\to\infty} \inf_{u\geqslant x} \frac{\mathbb{P}\left(S_n^\xi > u\right)}{\sum\limits_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\xi_k}(u)}$$

$$\geqslant \liminf_{x\to\infty} \frac{\mathbb{P}\left(S_n^\xi > x\right)}{\sum\limits_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\xi_k}(x)} \geqslant 1, \tag{4.30}$$

where the inequality (4.30) follows from Lemma 4.2. and now, by (4.29), this implies the lower estimate in (4.9). The Theorem 4.5 is proved. $\square$

4.4 Examples

4.4.1 Example 1

Let us consider three Pareto distributions: $\xi_1 \stackrel{d}{=} \mathcal{P}a_{\{2,1\}}$, $\xi_2 \stackrel{d}{=} \mathcal{P}a_{\{2,2\}}$, and $\xi_3 \stackrel{d}{=} \mathcal{P}a_{\{2,3\}}$, and let the generalised moment's function be $\varphi(x) = x/\log x$. Supposing that the r.v.s ξ_1, ξ_2, ξ_3 are pQAI, we derive the asymptotic formula for

$$\Delta(x) := \mathbb{E}\left(\varphi\left(S_3^\xi\right)\mathbb{I}_{\{S_3^\xi > x\}}\right) = \mathbb{E}\left(\left(\frac{\xi_1+\xi_2+\xi_3}{\log(\xi_1+\xi_2+\xi_3)}\right)\mathbb{I}_{\{\xi_1+\xi_2+\xi_3>x\}}\right).$$

We consider two separate cases regarding the dependence structure of the r.v.s ξ_1, ξ_2, ξ_3. In the first case the r.v.s are mutually independent, while in the second case we suppose that the random vector (ξ_1, ξ_2, ξ_3) follows the multivariate FGM distribution (see (2.29)):

$$\begin{aligned}\mathbb{P}(\xi_1 \leqslant x_1, \xi_2 \leqslant x_2, \xi_3 \leqslant x_3) &= F_{\xi_1}(x_1)F_{\xi_2}(x_2)F_{\xi_3}(x_3)\\ &\times\left(1 + \frac{1}{3}\overline{F}_{\xi_1}(x_1)\overline{F}_{\xi_2}(x_2) + \frac{1}{3}\overline{F}_{\xi_2}(x_2)\overline{F}_{\xi_3}(x_3) + \frac{1}{3}\overline{F}_{\xi_1}(x_1)\overline{F}_{\xi_3}(x_3)\right).\end{aligned}$$

It is easy to verify that all the conditions of Theorem 4.5 are satisfied. In addition, $L_{F_{\xi_1}} = L_{F_{\xi_2}} = L_{F_{\xi_3}}$. Therefore, according to Theorem 4.5, we obtain

$$\Delta(x) \underset{x\to\infty}{\sim} \sum_{k=1}^{3} \mathbb{E}\left(\frac{\xi_k}{\log \xi_k}\mathbb{I}_{\{\xi_k > x\}}\right).$$

Observe that

$$\begin{aligned}\mathbb{E}\left(\frac{\xi_k}{\log \xi_k}\mathbb{I}_{\{\xi_k > x\}}\right) &= 2k^2\int_x^\infty \frac{y}{\log y}\frac{\mathrm{d}y}{(k+y)^3} \underset{x\to\infty}{\sim} 2k^2\int_x^\infty \frac{\mathrm{d}y}{y^2\log y}\\ &= 2k^2\int_{\log x}^\infty \frac{\mathrm{e}^{-z}}{z}\mathrm{d}z \underset{x\to\infty}{\sim} \frac{2k^2}{x\log x}\end{aligned}$$

for large x and for all $k \in \{1, 2, 3\}$. It follows that

$$\Delta(x) \underset{x\to\infty}{\sim} \frac{28}{x\log x}.$$

The results of the simulated values of $\mathbb{E}(\varphi(S_n^\xi)\mathbb{I}_{\{S_n^\xi > x\}})$, together with the values of derived asymptotic formula (blue line), are presented in Figure 4.1.

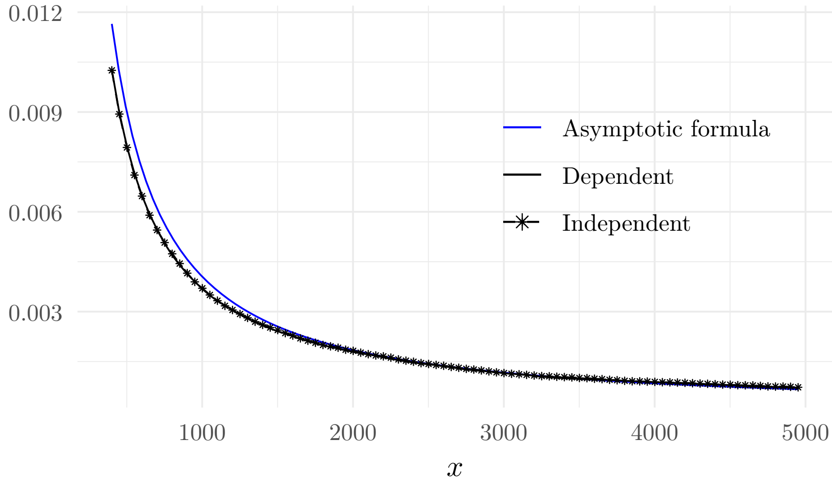

Figure 4.1: Simulated and asymptotic values for $\Delta(x)$ of Example 4.4.1.

4.4.2 Example 2

Let us consider two r.v.s distributed according to the Peter and Paul laws: $\xi_1 \overset{d}{=} \mathcal{PP}_{\{1,2\}}$, $\xi_2 \overset{d}{=} \mathcal{PP}_{\{1,\sqrt{2}\}}$, and let the generalised moment's function be $\varphi(x) = \sqrt[3]{x}\log x$. By supposing that the r.v.s ξ_1, ξ_2 are pQAI, we find the asymptotic formula for the truncated generalised moment

$$\widehat{\Delta}(x) := \mathbb{E}\left(\varphi(S_2^\xi)\mathbb{I}_{\{S_2^\xi > x\}}\right) = \mathbb{E}\left(\sqrt[3]{\xi_1+\xi_2}\log(\xi_1+\xi_2)\mathbb{I}_{\{\xi_1+\xi_2>x\}}\right).$$

We distinguish between independence and dependence cases of the r.v.s ξ_1, ξ_2. In the case of dependence, we suppose that the random vector (ξ_1, ξ_2) follows the distribution induced by the FGM copula (see (2.18))

$$\mathbb{P}\left(\xi_1 \leqslant x_1,\, \xi_2 \leqslant x_2\right) = F_{\xi_1}(x_1)F_{\xi_2}(x_2)\left(1 + \frac{1}{2}\overline{F}_{\xi_1}(x_1)\overline{F}_{\xi_2}(x_2)\right).$$

Since all the conditions of Theorem 4.5 are satisfied, we get

$$\begin{aligned}\widehat{\Delta}(x) &\underset{x\to\infty}{\gtrsim} \sum_{k=1}^{2} L_{F_{\xi_k}} \mathbb{E}\left(\sqrt[3]{\xi_k}\log(\xi_k)\mathbb{I}_{\{\xi_k>x\}}\right),\\ \widehat{\Delta}(x) &\underset{x\to\infty}{\lesssim} \sum_{k=1}^{2} L_{F_{\xi_k}}^{-1} \mathbb{E}\left(\sqrt[3]{\xi_k}\log(\xi_k)\mathbb{I}_{\{\xi_k>x\}}\right).\end{aligned} \tag{4.31}$$

To obtain the L indices for the r.v.s ξ_1 and ξ_2, observe that for a r.v. $\xi \overset{d}{=} \mathcal{PP}_{\{a,b\}}$ and $y > 1$,

$$\liminf_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} = \liminf_{x\to\infty} \left(b^{-a}\right)^{\lfloor \log_b y\rfloor + \lfloor \langle \log_b x\rangle + \langle \log_b y\rangle \rfloor} = \left(b^{-a}\right)^{\lfloor \log_b y\rfloor + 1}.$$

It then follows that

$$L_{F_\xi} = \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} = \lim_{y\downarrow 1} \left(b^{-a}\right)^{\lfloor \log_b y\rfloor + 1} = b^{-a}$$

and, as a result, $L_{F_{\xi_1}} = 1/2$ and $L_{F_{\xi_2}} = 1/\sqrt{2}$. According to the definition of Peter and Paul distribution,

$$\mathbb{P}(\xi_1 = 2^k) = 2^{-k}, \quad \mathbb{P}(\xi_2 = 2^{k/2}) = (\sqrt{2}-1)2^{-k/2}, \quad k \in \{1,2,\ldots\}.$$

Therefore,

$$\begin{aligned}\mathbb{E}\left(\sqrt[3]{\xi_1}\log\xi_1\mathbb{I}_{\{\xi_1>x\}}\right) &= \log 2 \sum_{k=\lfloor \log_2 x\rfloor+1}^{\infty} k\,2^{-2k/3}\\ &= \frac{\sqrt[3]{2}\log 2}{(2-\sqrt[3]{2})^2}\,2^{-2\lfloor \log_2 x\rfloor/3}\left(2 + (2-\sqrt[3]{2})\lfloor \log_2 x\rfloor\right),\end{aligned}$$

and, similarly,

$$\begin{aligned}\mathbb{E}\left(\sqrt[3]{\xi_2}\log\xi_2\mathbb{I}_{\{\xi_2>x\}}\right) &= \frac{(\sqrt{2}-1)\log 2}{2} \sum_{k=\lfloor 2\log_2 x\rfloor+1}^{\infty} k\,2^{-k/3}\\ &= \frac{(\sqrt{2}-1)\sqrt[3]{2}\log 2}{(2-\sqrt[3]{4})^2}\,2^{-\lfloor 2\log_2 x\rfloor/3}\\ &\quad\times\left(\sqrt[3]{2} + (\sqrt[3]{2}-1)\lfloor 2\log_2 x\rfloor\right).\end{aligned}$$

The last relations and the asymptotic estimate (4.31) imply that

$$\widehat{\Delta}(x) \underset{x\to\infty}{\lesssim} \sqrt[3]{2}\log 2 \left(\frac{2}{(2-\sqrt[3]{2})^2}\, 2^{-2\lfloor \log_2 x\rfloor/3}\big(2+(2-\sqrt[3]{2})\lfloor \log_2 x\rfloor\big) \right.$$
$$\left. + \frac{2-\sqrt{2}}{(2-\sqrt[3]{4})^2}\, 2^{-\lfloor 2\log_2 x\rfloor/3}\big(\sqrt[3]{2}+(\sqrt[3]{2}-1)\lfloor 2\log_2 x\rfloor\big) \right),$$
$$\widehat{\Delta}(x) \underset{x\to\infty}{\gtrsim} \sqrt[3]{2}\log 2 \left(\frac{1}{2(2-\sqrt[3]{2})^2}\, 2^{-2\lfloor \log_2 x\rfloor/3}\big(2+(2-\sqrt[3]{2})\lfloor \log_2 x\rfloor\big) \right.$$
$$\left. + \frac{\sqrt{2}-1}{\sqrt{2}(2-\sqrt[3]{4})^2}\, 2^{-\lfloor 2\log_2 x\rfloor/3}\big(\sqrt[3]{2}+(\sqrt[3]{2}-1)\lfloor 2\log_2 x\rfloor\big) \right).$$

The results of the simulated values of $\mathbb{E}(\varphi(S_n^\xi)\mathbb{I}_{\{S_n^\xi > x\}})$, together with the values of the derived asymptotic formula, are presented in Figure 4.2.

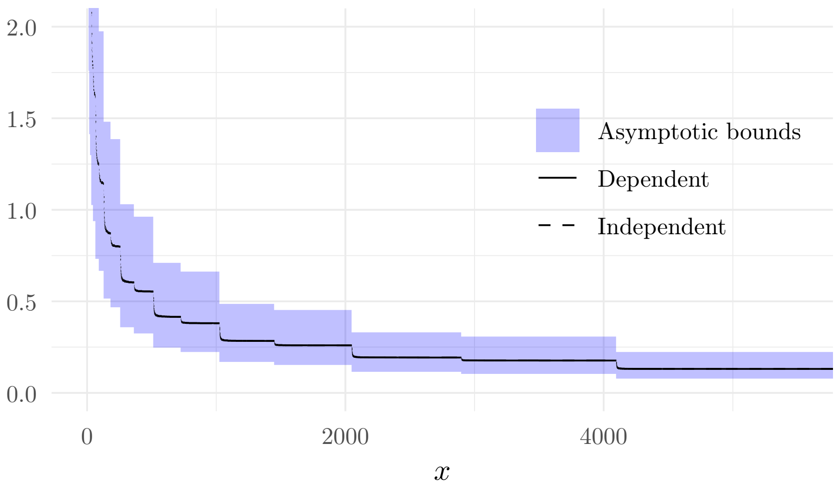

Figure 4.2: Simulated and asymptotic values for $\widehat{\Delta}(x)$ of Example 4.4.2.

4.4.3 Example 3

Let us consider two r.v.s distributed according to the Cai-Tang law with parameter $p = 1/2$, i.e., $\xi_1 \overset{d}{=} \xi_2 \overset{d}{=} \mathcal{CT}_{1/2}$. In addition, we suppose that the random weights are independent and discrete uniformly distributed with different parameters, i.e.,

$$\mathbb{P}(\theta_1 = 0) = \mathbb{P}(\theta_1 = 1) = \frac{1}{2},$$
$$\mathbb{P}(\theta_2 = 0) = \mathbb{P}(\theta_2 = 1) = \mathbb{P}(\theta_2 = 2) = \frac{1}{3}.$$

Supposing that the r.v.s ξ_1, ξ_2 are pQAI and the pairs $\{\xi_1, \xi_2\}$, $\{\theta_1, \theta_2\}$ are independent, we derive the asymptotic formula for

$$\widetilde{\Delta}(x) := \mathbb{E}\left(\sqrt{\theta_1\xi_1 + \theta_2\xi_2}\,\log(\theta_1\xi_1 + \theta_2\xi_2)\mathbb{I}_{\{\theta_1\xi_1+\theta_2\xi_2>x\}}\right).$$

As in the previous examples in this section, we consider both dependence and independence of the r.v.s ξ_1, ξ_2. In the case of dependence, we suppose that the random vector (ξ_1, ξ_2) follows the dependence structure implied by the AMH copula with parameter $\vartheta = 0.8$:

$$\mathbb{P}(\xi_1 \leqslant x_1,\, \xi_2 \leqslant x_2) = \frac{F_{\xi_1}(x_1)F_{\xi_2}(x_2)}{1 - 0.8\overline{F}_{\xi_1}(x_1)\overline{F}_{\xi_2}(x_2)}.$$

In the case under consideration, we have $\varphi(x) = \sqrt{x}\log x$ and by (2.3), for all $x \geqslant 1$ we obtain

$$\overline{F}_{\xi_1}(x) = \overline{F}_{\xi_2}(x) = \frac{1}{2}\,2^{-\lfloor \log_2 x\rfloor}\left(3 - x\,2^{-\lfloor \log_2 x\rfloor}\right).$$

Since $L_{F_{\xi_1}} = L_{F_{\xi_2}} = 1$, it follows from Theorem 4.6 that

$$\begin{aligned}\widetilde{\Delta}(x) \underset{x\to\infty}{\sim} \mathbb{E}&\left(\sqrt{\theta_1\xi_1}\,\log(\theta_1\xi_1)\mathbb{I}_{\{\theta_1\xi_1>x\}}\right)\\ &+\mathbb{E}\left(\sqrt{\theta_2\xi_2}\,\log(\theta_2\xi_2)\mathbb{I}_{\{\theta_2\xi_2>x\}}\right).\end{aligned} \tag{4.32}$$

Using Lemma 4.1, for $x \in [2^K, 2^{K+1}]$, $K \in \mathbb{N}$, we have

$$\begin{aligned}\mathbb{E}\left(\sqrt{\theta_1\xi_1}\,\log(\theta_1\xi_1)\mathbb{I}_{\{\theta_1\xi_1>x\}}\right) &= \frac{1}{2}\mathbb{E}\left(\sqrt{\xi_1}\,\log(\xi_1)\mathbb{I}_{\{\xi_1>x\}}\right)\\ &= \frac{1}{2}\left(\sqrt{x}\log x\,\overline{F}_{\xi_1}(x) + \int_x^\infty \frac{\log y + 2}{2\sqrt{y}}\,\overline{F}_{\xi_1}(y)\,\mathrm{d}y\right)\\ &= \frac{\sqrt{x}\log x}{4}2^{-\lfloor \log_2 x\rfloor}\left(3 - x\,2^{-\lfloor \log_2 x\rfloor}\right) - \mathcal{I}_1 + \mathcal{I}_2,\end{aligned}$$

where

$$
\begin{aligned}
\mathcal{I}_1 &= \frac{1}{8}\frac{1}{2^K}\int_{2^K}^{x} \frac{\log y + 2}{\sqrt{y}}\left(3 - \frac{y}{2^K}\right)\mathrm{d}y \\
&= \frac{3}{4}\frac{\sqrt{x}\log x}{2^K} - \frac{1}{12}\frac{\sqrt{x^3}\log x}{2^{2K}} - \frac{1}{9}\frac{\sqrt{x^3}}{2^{2K}} - \frac{2}{3}\frac{K\log 2}{\sqrt{2^K}} + \frac{1}{9}\frac{1}{\sqrt{2^K}}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_2 &= \frac{1}{8}\sum_{m=K}^{\infty}\int_{2^m}^{2^{m+1}} \frac{\log y + 2}{\sqrt{y}}\frac{1}{2^m}\left(3 - \frac{y}{2^m}\right)\mathrm{d}y \\
&= \frac{3}{8}\sum_{m=K}^{\infty}\frac{1}{2^m}\int_{2^m}^{2^{m+1}} \frac{\log y + 2}{\sqrt{y}}\,\mathrm{d}y \\
&\qquad - \frac{1}{8}\sum_{m=K}^{\infty}\frac{1}{2^{2m}}\int_{2^m}^{2^{m+1}} (\log y + 2)\sqrt{y}\,\mathrm{d}y \\
&= \frac{\begin{array}{l} 3(3\sqrt{2}+1)K^2\log 2 + (42\sqrt{2}\log 2 + 48\log 2 - 6\sqrt{2} - 4)K \\ \qquad + 90\sqrt{2}\log 2 + 126\log 2 - 20 - 16\sqrt{2} \end{array}}{18(K+\sqrt{2}+2)\sqrt{2^K}}.
\end{aligned}
$$

Bearing in mind that $K = \lfloor \log_2 x \rfloor$, from the above equalities we derive

$$
\begin{aligned}
\mathbb{E}\left(\sqrt{\theta_1\xi_1}\log(\theta_1\xi_1)\mathbb{I}_{\{\theta_1\xi_1 > x\}}\right) \underset{x\to\infty}{\sim} &\left(\frac{5}{6} + \frac{\sqrt{2}}{2}\right)\frac{\lfloor \log_2 x\rfloor \log 2}{2^{\lfloor \log_2 x\rfloor/2}} \\
&- \frac{1}{6}\frac{\sqrt{x^3}\log x}{2^{2\lfloor \log_2 x\rfloor}}.
\end{aligned} \tag{4.33}
$$

For the second term in (4.32), we have

$$
\begin{aligned}
\mathbb{E}\left(\sqrt{\theta_2\xi_2}\log(\theta_2\xi_2)\mathbb{I}_{\{\theta_2\xi_2 > x\}}\right) = &\frac{1}{3}\mathbb{E}\left(\sqrt{\xi_2}\log(\xi_2)\mathbb{I}_{\{\xi_2 > x\}}\right) \\
&+ \frac{\sqrt{2}}{3}\mathbb{E}\left(\sqrt{\xi_2}\log(\xi_2)\mathbb{I}_{\{\xi_2 > x/2\}}\right) \\
&+ \frac{\sqrt{2}\log 2}{3}\mathbb{E}\left(\sqrt{\xi_2}\mathbb{I}_{\{\xi_2 > x/2\}}\right).
\end{aligned}
$$

Using the derived asymptotic equality (4.33), we get

$$
\begin{aligned}
\frac{1}{3}\mathbb{E}\left(\sqrt{\xi_2}\log(\xi_2)\mathbb{I}_{\{\xi_2 > x\}}\right) \underset{x\to\infty}{\sim} &\left(\frac{5}{9} + \frac{\sqrt{2}}{3}\right)\frac{\lfloor \log_2 x\rfloor \log 2}{2^{\lfloor \log_2 x\rfloor/2}} \\
&- \frac{1}{9}\frac{\sqrt{x^3}\log x}{2^{2\lfloor \log_2 x\rfloor}},
\end{aligned}
$$

$$\frac{\sqrt{2}}{3}\mathbb{E}\left(\sqrt{\xi_2}\log(\xi_2)\mathbb{I}_{\{\xi_2>x/2\}}\right) \underset{x\to\infty}{\sim} \left(\frac{10}{9}+\frac{2\sqrt{2}}{3}\right)\frac{\lfloor\log_2 x\rfloor\log 2}{2^{\lfloor\log_2 x\rfloor/2}} - \frac{\sqrt{2}}{18}\frac{\sqrt{x^3}\log x}{2^{2\lfloor\log_2 x\rfloor}}.$$

In addition,

$$\mathbb{E}\left(\sqrt{\xi_2}\mathbb{I}_{\{\xi_2>x/2\}}\right) = O\left(\frac{1}{\log x}\mathbb{E}\left(\sqrt{\xi_2}\log(\xi_2)\mathbb{I}_{\{\xi_2>x/2\}}\right)\right).$$

Consequently,

$$\mathbb{E}\left(\sqrt{\theta_2\xi_2}\log(\theta_2\xi_2)\mathbb{I}_{\{\theta_2\xi_2>x\}}\right) \underset{x\to\infty}{\sim} \left(\frac{5}{3}+\sqrt{2}\right)\frac{\lfloor\log_2 x\rfloor\log 2}{2^{\lfloor\log_2 x\rfloor/2}} - \frac{1}{9}\left(1+\frac{\sqrt{2}}{2}\right)\frac{\sqrt{x^3}\log x}{2^{2\lfloor\log_2 x\rfloor}} \tag{4.34}$$

and, due to the estimates (4.32), (4.33) and (4.34),

$$\widetilde{\Delta}(x) \underset{x\to\infty}{\sim} \left(\frac{5}{2}+\frac{3\sqrt{2}}{2}\right)\frac{\lfloor\log_2 x\rfloor\log 2}{2^{\lfloor\log_2 x\rfloor/2}} - \frac{1}{18}(5+\sqrt{2})\frac{\sqrt{x^3}\log x}{2^{2\lfloor\log_2 x\rfloor}}.$$

The simulated values of $\mathbb{E}(\varphi(S_n^\xi)\mathbb{I}_{\{S_n^\xi>x\}})$, together with the values of the derived asymptotic formula (blue line), are presented in Figure 4.3.

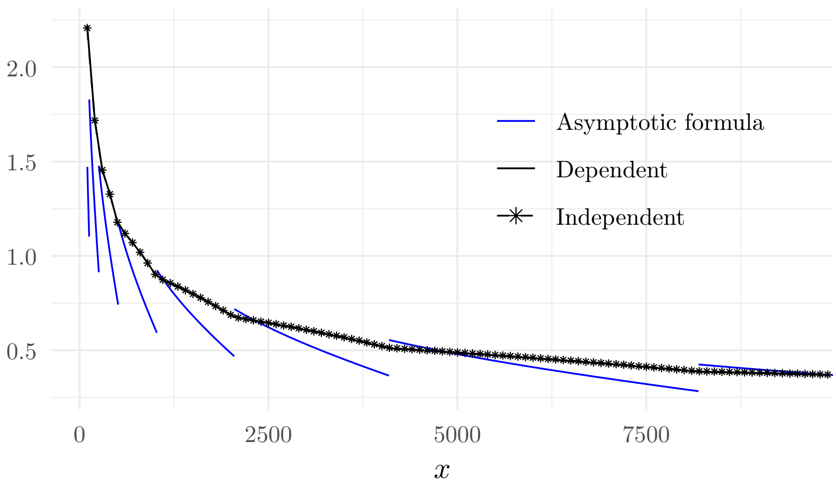

Figure 4.3: Simulated and asymptotic values for the $\widetilde{\Delta}(x)$ of Example 4.4.3.

5 Asymptotic behaviour of the Haezendonck-Goovaerts risk measure

5.1 Related results

Despite its well-established theoretical properties, such as coherence (see Proposition 2.4), and its greater flexibility compared to the alternative ES risk measure, HG risk measure generally does not admit an analytical expression, since an explicit solution to the equation (2.42) is not attainable [108]. However, in certain cases analytical expressions and asymptotic formulas can be derived. One such case occurs when the normalised Young function is a power function $\varphi(t) = t^{\varkappa}$ with $\varkappa \geqslant 1$ (see [108, 80]).

In particular, Tang and Yang in [108] obtained the following expression of the HG risk measure which motivated the research presented in this dissertation.

Theorem 5.1 (See [108, Theorem 2.1 and (1.3)]). *Consider the power Young function $\varphi(t) = t^{\varkappa}$ with $\varkappa \geqslant 1$.*

(i) *If $q \in (0,1)$, $\varkappa = 1$, and $\mathbb{E}\xi^+ < \infty$, then*

$$\mathrm{HG}_q(\xi) = F_\xi^{\leftarrow}(q) + \frac{\mathbb{E}\Big(\xi - F_\xi^{\leftarrow}(q)\Big)^+}{1-q}. \tag{5.1}$$

(ii) *If $q \in (0,1)$, $\varkappa > 1$, $\mathbb{P}(\xi = F_\xi^{\leftarrow}(1)) = 0$, and $\mathbb{E}(\xi^+)^{\varkappa} < \infty$, then*

$$\mathrm{HG}_q(\xi) = x + \left(\frac{\mathbb{E}\big((\xi - x)^+\big)^{\varkappa}}{1-q}\right)^{1/\varkappa}, \tag{5.2}$$

where $x = x(q) \in (-\infty, F_\xi^{\leftarrow}(1))$ is the unique solution of the equation

$$\frac{\Big(\mathbb{E}\big((\xi - x)^+\big)^{\varkappa-1}\Big)^{\varkappa}}{\Big(\mathbb{E}\big((\xi - x)^+\big)^{\varkappa}\Big)^{\varkappa-1}} = 1 - q. \tag{5.3}$$

In the same paper Tang and Yang also investigated a special case when a random risk ξ belongs to the class of regularly varying functions. It was shown, that in this setting, the tail of the HG risk measure is asymptotically equivalent to a specific function involving Beta function and a generalised quantile function $F_\xi^{\leftarrow}$.

Theorem 5.2 (See [108, Theorem 4.1]). *Let the power Young function $\varphi(t) = t^{\varkappa}$ with $\varkappa \geqslant 1$, and let ξ be a real-valued r.v. such that $F_\xi \in \mathscr{R}_\alpha$ for some $\alpha > \varkappa$. Then*

$$\mathrm{HG}_q(\xi) \underset{q\uparrow 1}{\sim} \frac{\alpha(\alpha-\varkappa)^{(\varkappa/\alpha)-1}}{\varkappa^{(\varkappa-1)/\alpha}} \big(B(\alpha-\varkappa,\varkappa)\big)^{1/\alpha} F_\xi^{\leftarrow}(q),$$

where

$$B(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}\mathrm{d}y, \quad a > 0,\ b > 0,$$

is the beta function.

We also note another study by Tang and Yuan [109] in which the asymptotics of the HG risk measure is further explored in a more general setting, namely, for a broader subclass of Young functions within the framework of extreme value theory. However, the literature on the analytical and asymptotic properties of the measure remains relatively scarce.

5.2 Main results

Inspired by Theorem 5.1 obtained in [108], we investigate further the asymptotic behaviour of the HG measure. In this subsection, we state two theorems concerning the asymptotic behaviour of the HG risk measure for the sum of consistently varying loss-profit distributions, when the normalised Young function takes the form of a power function $\varphi(t) = t^{\varkappa}$, $\varkappa \geqslant 1$. The theorems establish that, when the confidence level q of the risk measure tends to 1, the HG risk measure of a random sum admits an asymptotic representation similar to that given in Theorem 5.2 without the need to estimate complicated distributions of random sums with not necessarily independent random summands. The following theorem deals with the random losses of the form S_n^ξ and shows that the asymptotics of the HG risk measure only depend on consistently varying summands and not on possibly lighter tailed increments.

Theorem 5.3 (See [98, Theorem 6]). *Consider the power Young function $\varphi(t) = t^\varkappa$ with $\varkappa \geqslant 1$, and let $\{\xi_1, \ldots, \xi_n\}$ be a collection of real-valued* pQAI *r.v.s.*

(i) *Let us suppose that $\varkappa = 1$, $\max_{1\leqslant k\leqslant n} \mathbb{E}(\xi_k^+) < \infty$, $F_{\xi_L} \in \mathscr{C}$ for some $L \in \{1, \ldots, n\}$, and for the other indices $k \neq L$ let $F_{\xi_k} \in \mathscr{C}$ or $\mathbb{P}(|\xi_k| > x) = o(\overline{F}_{\xi_L}(x))$. If $F_{\xi_k} \in \mathscr{P}_\mathscr{D}$ for any index $k \in \mathcal{I}_n = \{k \in \{1, \ldots, n\} : F_{\xi_k} \in \mathscr{C}\}$, then*

$$\mathrm{HG}_q\Big(S_n^\xi\Big) \underset{q\uparrow 1}{\sim} H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+,$$

where $H_n^{\leftarrow}$ is the quantile function of the d.f.

$$H_n := \max\Big\{0, 1 - \sum_{k\in\mathcal{I}_n} \overline{F}_{\xi_k}\Big\}.$$

(ii) *Let us suppose that $\varkappa \geqslant 2$, $\max_{1\leqslant k\leqslant n} \mathbb{E}(\xi_k^+)^\varkappa < \infty$, $F_{\xi_L} \in \mathscr{C}$ for some $L \in \{1, \ldots, n\}$, and for the other indices $k \neq L$ let $F_{\xi_k} \in \mathscr{C}$ or $\mathbb{P}(|\xi_k| > x) = o(\overline{F}_{\xi_L}(x))$. In addition, let*

$$\left(\max_{k\in\mathcal{I}_n} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(yx)}{\overline{F}_{\xi_k}(x)}\right)^\varkappa \left(\max_{k\in\mathcal{I}_n} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_k}(yx)}\right)^{\varkappa-1} < 1 \quad (5.4)$$

for some $y > 1$, where $\mathcal{I}_n = \{k \in \{1, \ldots, n\} : F_{\xi_k} \in \mathscr{C}\}$. Then

$$\mathrm{HG}_q\Big(S_n^\xi\Big) \underset{q\uparrow 1}{\sim} \widehat{x}(q) + \left(\frac{1}{1-q}\sum_{k\in\mathcal{I}_n} \mathbb{E}\Big((\xi_k - \widehat{x}(q))^+\Big)^\varkappa\right)^{1/\varkappa},$$

where $\widehat{x} = \widehat{x}(q)$ *is the unique solution to the equation*

$$\frac{\Big(\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big((\xi_k-\widehat{x})^+\big)^{\varkappa-1}\Big)\Big)^{\varkappa}}{\Big(\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big((\xi_k-\widehat{x})^+\big)^{\varkappa}\Big)\Big)^{\varkappa-1}} = 1-q.$$

The second theorem given below is a generalisation of Theorem 5.3 to the case of randomly weighted sums. This time, however, we make a stricter assumption that not some, but all primary random variables belong to the class $\mathscr{C}$.

Theorem 5.4 ([98, Theorem 7])**.** *Consider the power Young function* $\varphi(t) = t^{\varkappa}$ *with* $\varkappa \geqslant 1$ *and let* $\{\xi_1,\dots,\xi_n\}$ *be a collection of real-valued* pQAI *r.v.s such that* $F_{\xi_k} \in \mathscr{C}$ *for all* $k = 1,\dots,n$. *Let* $\{\theta_1,\dots,\theta_n\}$ *be another collection of arbitrary dependent, nonnegative, and nondegenerate at zero r.v.s. Suppose that the collections* $\{\xi_1,\dots,\xi_n\}$ *and* $\{\theta_1,\dots,\theta_n\}$ *are independent and* $\max_{1\leqslant k\leqslant n}\{\mathbb{E}\theta_k^p\}$ *is finite for some* $p > \max_{1\leqslant k\leqslant n}\{J^+_{F_{\xi_k}}\}$.

(i) *If* $\varkappa = 1$, $\max\limits_{1\leqslant k\leqslant n}\mathbb{E}(\xi_k^+) < \infty$, *and* $F_{\xi_k} \in \mathscr{P}_{\mathscr{D}}$ *for each* $k \in \{1,\dots,n\}$, *then*

$$\mathrm{HG}_q\Big(S_n^{\theta\xi}\Big) \underset{q\uparrow 1}{\sim} H_n^{*\leftarrow}(q) + \frac{1}{1-q}\sum_{k=1}^{n}\mathbb{E}\Big(\theta_k\xi_k - H_n^{*\leftarrow}(q)\Big)^+,$$

where $H_n^{*\leftarrow}$ *is the quantile function of the d.f.*

$$H_n^* := \max\Big\{0, 1-\sum_{k=1}^{n}\overline{F}_{\theta_k\xi_k}\Big\}.$$

(ii) *Let* $\varkappa \geqslant 2$, $\max\limits_{1\leqslant k\leqslant n}\mathbb{E}(\xi_k^+)^{\varkappa} < \infty$. *Let, in addition,*

$$\left(\max_{1\leqslant k\leqslant n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(yx)}{\overline{F}_{\xi_k}(x)}\right)^{\varkappa}\left(\max_{1\leqslant k\leqslant n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_k}(yx)}\right)^{\varkappa-1} < 1$$

for some $y > 1$. *Then*

$$\mathrm{HG}_q\Big(S_n^{\theta\xi}\Big) \underset{q\uparrow 1}{\sim} x^*(q) + \left(\frac{1}{1-q}\sum_{k=1}^{n}\mathbb{E}\Big(\big(\theta_k\xi_k - x^*(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa},$$

where $x^* = x^*(q)$ is the unique solution to the equation

$$\frac{\Big(\sum_{k=1}^n \mathbb{E}\Big(\big(\theta_k\xi_k - x^*\big)^+\Big)^{\varkappa-1}\Big)^{\varkappa}}{\Big(\sum_{k=1}^n \mathbb{E}\Big(\big(\theta_k\xi_k - x^*\big)^+\Big)^{\varkappa}\Big)^{\varkappa-1}} = 1 - q.$$

We remark that Theorems 5.3 and 5.4 do not cover the case $\varkappa \in (1, 2)$ that remains an open problem for further research.

5.3 Proofs of the main results

5.3.1 Auxiliary assertions

Before proceeding with the proofs of Theorems 5.3 and 5.4 we need several auxiliary statements. These fall into three groups—properties of quantile functions of consistently varying or positively decreasing distributions, characteristics of the function on the right hand side of the equation 5.3, and certain closure properties of primary random variables and random weights.

Properties of quantile functions

We begin with a couple of lemmas (see Lemmas 5.1 and 5.2 below) which extend parts (3)-(4) of Lemma 2.2 to an asymptotic setting with either consistently varying or positively decreasing distributions.

Lemma 5.1. *Let F be a d.f. from the class $\mathscr{C}$. Then*

$$F\Big(F^{\leftarrow}(q)\Big) \underset{q\uparrow 1}{\sim} q, \quad \text{and} \quad \overline{F}\Big(\overline{F}^{\leftarrow}(p)\Big) \underset{p\downarrow 0}{\sim} p.$$

Proof. Let $q \in (0, 1)$. By part (4) of Lemma 2.2 if follows

$$F\Big(F^{\leftarrow}(q)\Big) \geqslant q. \tag{5.5}$$

Therefore,

$$F\Big(F^{\leftarrow}(q)\Big) \underset{q\uparrow 1}{\gtrsim} q. \tag{5.6}$$

For the first asymptotic relation of the lemma, it remains to prove that

$$\limsup_{q\uparrow 1} \frac{F\Big(F^{\leftarrow}(q)\Big)}{q} \leqslant 1. \tag{5.7}$$

Suppose, on the contrary, that

$$\limsup_{q\uparrow 1} \frac{F\Big(F^{\leftarrow}(q)\Big)}{q} > 1.$$

Hence, there exist $\Delta > 0$ and a sequence $q_n \uparrow 1$ such that

$$\lim_{n\to\infty} \frac{F\Big(F^{\leftarrow}(q_n)\Big)}{q_n} = 1 + \Delta,$$

implying that $F\Big(F^{\leftarrow}(q_n)\Big) > (1 + \Delta/2)q_n$ for large enough n. Since $F(x) \leqslant 1$ for any x, we have

$$q_n < \big(1 + \Delta/2\big)^{-1}$$

for large n. This is impossible because $q_n \uparrow 1$. The contradiction obtained shows that the estimate (5.7) holds. The estimates (5.6) and (5.7) imply the first asymptotic relation of the lemma.

Now, let us consider the second relation of the lemma. Let $p > 0$. By case (4) of Lemma 2.2, we have

$$\overline{F}\left(\overline{F}^{\leftarrow}(p)\right) \leqslant p,$$

implying that

$$\overline{F}\Big(\overline{F}^{\leftarrow}(p)\Big) \underset{p\downarrow 0}{\lesssim} p. \tag{5.8}$$

It is left to prove that

$$\liminf_{p\downarrow 0} \frac{\overline{F}\Big(\overline{F}^{\leftarrow}(p)\Big)}{p} \geqslant 1. \tag{5.9}$$

Let us suppose, on the contrary, that

$$\liminf_{p \downarrow 0} \frac{\overline{F}\left(\overline{F}^{\leftarrow}(p)\right)}{p} < 1.$$

It follows that there exist $\Delta \in (0,1)$ and a sequence $p_n \downarrow 0$ satisfying

$$\lim_{n \to \infty} \frac{\overline{F}\left(\overline{F}^{\leftarrow}(p_n)\right)}{p_n} = 1 - \Delta.$$

Hence,

$$\overline{F}\left(\overline{F}^{\leftarrow}(p_n)\right) < (1 - \Delta/2)p_n,$$

or, equivalently, by (2.33),

$$F\big(\overline{F}^{\leftarrow}(p_n)\big) = F\big(F^{\leftarrow}(1 - p_n)\big) > 1 - p_n + p_n(\Delta/2),$$

for large enough n.

Let us temporarily denote

$$x_n := F^{\leftarrow}(1 - p_n).$$

Since $\overline{F}(x) > 0$ for any $x \in \mathbb{R}$, the sequence x_n is nondecreasing and $x_n \uparrow \infty$. In addition, F is continuous from the right, so we have $F(x_n) \geqslant 1 - p_n$. For a fixed n, there are only two possible cases: $F(x_n) = 1 - p_n$ or $F(x_n) > 1 - p_n$. In the first case, we get

$$1 - p_n > 1 - p_n + p_n(\Delta/2),$$

which is impossible since Δ is positive. Consequently, F has a jump at each point x_n. Thus, it follows that

$$F\big((1 - \varepsilon_n)x_n\big) < 1 - p_n \quad \text{and} \quad F(x_n) > (1 - p_n) + p_n(\Delta/2),$$

or, equivalently,

$$\overline{F}\big((1 - \varepsilon_n)x_n\big) > p_n \quad \text{and} \quad \overline{F}(x_n) < p_n(1 - \Delta/2),$$

for some sequence $\varepsilon_n \downarrow 0$. These both estimates imply that

$$\lim_{n\to\infty} \frac{\overline{F}(((1-\varepsilon_n)x_n)}{\overline{F}(x_n)} \geqslant \frac{2}{2-\Delta} > 1$$

for some sequences $x_n \uparrow \infty$ and $\varepsilon_n \downarrow 0$. The derived relation contradicts to the condition $F \in \mathscr{C}$ because it should be

$$\lim_{\varepsilon\downarrow 0} \limsup_{x\to\infty} \frac{\overline{F}((1-\varepsilon)x)}{\overline{F}(x)} \leqslant 1,$$

according to Definition 2.4.

The contradiction obtained proves inequality (5.9), which, together with the estimate (5.8), proves the second asymptotic relation of the lemma. $\square$

Lemma 5.2. *Let F be a d.f. from the class $\mathscr{P}_{\mathscr{D}}$. Then*

$$\overline{F}^{\leftarrow}\big(\overline{F}(x)\big) \underset{x\to\infty}{\sim} x, \quad \text{and} \quad F^{\leftarrow}\big(F(x)\big) \underset{x\to\infty}{\sim} x.$$

Proof. Let us consider the first relation. According to the definition of the quantile function $\overline{F}^{\leftarrow}$ (see Definition 2.15), we have

$$\overline{F}^{\leftarrow}\big(\overline{F}(x)\big) = \inf\{y : \overline{F}(y) \leqslant \overline{F}(x)\}.$$

By the case (3) of Lemma 2.2, we have $\overline{F}^{\leftarrow}\big(\overline{F}(x)\big) \leqslant x$, implying

$$\overline{F}^{\leftarrow}\big(\,\overline{F}(x)\big) \underset{x\to\infty}{\lesssim} x. \tag{5.10}$$

It remains to prove that

$$\overline{F}^{\leftarrow}\big(\,\overline{F}(x)\big) \underset{x\to\infty}{\gtrsim} x. \tag{5.11}$$

Assume, on the contrary, that

$$\liminf_{x\to\infty} \frac{\overline{F}^{\leftarrow}\big(\,\overline{F}(x)\big)}{x} < 1.$$

In such a case, there exist $\Delta \in (0{,}1)$ and a sequence $x_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \frac{\overline{F}^{\leftarrow}\big(\,\overline{F}(x_n)\big)}{x_n} = 1 - \Delta.$$

Hence,

$$\overline{F}^{\leftarrow}\left(\overline{F}(x_n)\right) = \inf\{y : \overline{F}(y) \leqslant \overline{F}(x_n)\} \leqslant (1 - \Delta/2)x_n$$

for sufficiently large n, say $n \geqslant N_\Delta$. This relation implies that $\overline{F}((1 - \Delta/4)x_n) \leqslant \overline{F}(x_n)$, while $\overline{F}$ being nonincreasing ensures that $\overline{F}((1 - \Delta/4)x_n) \geqslant \overline{F}(x_n)$. As a result,

$$\overline{F}\left(\left(1 - \frac{\Delta}{4}\right)x_n\right) = \overline{F}(x_n)$$

if $n \geqslant N_\Delta$. It follows that

$$\lim_{n\to\infty} \frac{\overline{F}\left(\left(1 - \frac{\Delta}{4}\right)x_n\right)}{\overline{F}(x_n)} = 1,$$

which contradicts to the definition of $F \in \mathscr{P}_{\mathscr{D}}$ (see Definition 2.8). Consequently, the asymptotic relation (5.11) holds. The relations (5.10) and (5.11) imply the first relation of the lemma.

The second relation of the lemma follows from the following equalities:

$$\lim_{x\to\infty} \frac{F^{\leftarrow}(F(x))}{x} = \lim_{x\to\infty} \frac{F^{\leftarrow}(1 - \overline{F}(x))}{x} = \lim_{x\to\infty} \frac{\overline{F}^{\leftarrow}\left(\overline{F}(x)\right)}{x} = 1, \quad (5.12)$$

where the second equality in (5.12) is due to (2.33). Lemma 5.2 is now proved. □

The following lemma shows, that if two positive functions are asymptotically equivalent and decaying to zero, the equivalence relation is preserved under the generalised inverse transformation of a positively decreasing tail function.

Lemma 5.3. *Let F be a d.f. from the class $\mathscr{P}_{\mathscr{D}}$. Then for any two positive functions $a(x)$ and $b(x)$ such that $a(x) \sim b(x)$, $a(x) \to 0$, as $x \to \infty$, it holds that*

$$\overline{F}^{\leftarrow}(a(x)) \underset{x\to\infty}{\sim} \overline{F}^{\leftarrow}(b(x)),$$

and for any positive functions $\hat{a}(p)$, $\hat{b}(p)$, such that $\hat{a}(p) \sim \hat{b}(p)$ and

$\hat{a}(p) \to 0$ as $p \downarrow 0$, it holds that

$$\overline{F}^{\leftarrow}\left(\hat{a}(p)\right) \underset{p\downarrow 0}{\sim} \overline{F}^{\leftarrow}\left(\hat{b}(p)\right).$$

Proof. It is enough to prove the first equivalence relation because the second relation follows from the first by noting that

$$\lim_{p\downarrow 0} \frac{\overline{F}^{\leftarrow}\left(\hat{a}(p)\right)}{\overline{F}^{\leftarrow}\left(\hat{b}(p)\right)} = \lim_{x\to\infty} \frac{\overline{F}^{\leftarrow}\left(\hat{a}\left(\frac{1}{x}\right)\right)}{\overline{F}^{\leftarrow}\left(\hat{b}\left(\frac{1}{x}\right)\right)}.$$

Let us consider the first relation. First, we will show that

$$\liminf_{x\to\infty} \frac{\overline{F}^{\leftarrow}\left(a(x)\right)}{\overline{F}^{\leftarrow}\left(b(x)\right)} \geqslant 1. \tag{5.13}$$

We will proceed with the proof by contradiction again. Assume

$$\liminf_{x\to\infty} \frac{\overline{F}^{\leftarrow}\left(a(x)\right)}{\overline{F}^{\leftarrow}\left(b(x)\right)} < 1.$$

This means that there exist $\Delta \in (0,1)$ and a sequence $x_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \frac{\overline{F}^{\leftarrow}\left(a(x_n)\right)}{\overline{F}^{\leftarrow}\left(b(x_n)\right)} = 1 - \Delta.$$

It follows that

$$\frac{\overline{F}^{\leftarrow}\left(a(x_n)\right)}{\left(1-\frac{\Delta}{2}\right)} < \overline{F}^{\leftarrow}\left(b(x_n)\right)$$

for large enough n. By the definition of $\overline{F}^{\leftarrow}$,

$$\overline{F}\left(\frac{\overline{F}^{\leftarrow}\left(a(x_n)\right)}{\left(1-\frac{\Delta}{2}\right)}\right) > b(x_n).$$

This relation together with the case (4) of Lemma 2.2 implies

$$\frac{\overline{F}\left(\left(1-\frac{\Delta}{2}\right)\frac{\overline{F}^{\leftarrow}(a(x_n))}{\left(1-\frac{\Delta}{2}\right)}\right)}{\overline{F}\left(\frac{\overline{F}^{\leftarrow}(a(x_n))}{\left(1-\frac{\Delta}{2}\right)}\right)} = \frac{\overline{F}\left(\overline{F}^{\leftarrow}\left(a(x_n)\right)\right)}{\overline{F}\left(\frac{\overline{F}^{\leftarrow}(a(x_n))}{\left(1-\frac{\Delta}{2}\right)}\right)} \leqslant \frac{a(x_n)}{b(x_n)} \to 1$$

as $n \to \infty$, since $a(x) \sim b(x)$ as $x \to \infty$. Thus,

$$\lim_{n\to\infty} \frac{\overline{F}\left(\left(1-\frac{\Delta}{2}\right)\frac{\overline{F}^{\leftarrow}(a(x_n))}{\left(1-\frac{\Delta}{2}\right)}\right)}{\overline{F}\left(\frac{\overline{F}^{\leftarrow}(a(x_n))}{\left(1-\frac{\Delta}{2}\right)}\right)} \leqslant 1$$

for some sequence $\overline{F}^{\leftarrow}(a(x_n))/(1-\Delta/2) \to \infty$ as $n \to \infty$. This contradicts to the definition of $F \in \mathscr{P}_{\mathscr{D}}$ (see Definition 2.8).

It remains to prove

$$\limsup_{x\to\infty} \frac{\overline{F}^{\leftarrow}(a(x))}{\overline{F}^{\leftarrow}(b(x))} \leqslant 1.$$

Assume, on the contrary, that

$$\limsup_{x\to\infty} \frac{\overline{F}^{\leftarrow}(a(x))}{\overline{F}^{\leftarrow}(b(x))} > 1.$$

It follows that there exist $\Delta > 0$ and a sequence $x_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \frac{\overline{F}^{\leftarrow}(a(x_n))}{\overline{F}^{\leftarrow}(b(x_n))} = 1 + \Delta.$$

Consequently,

$$\overline{F}^{\leftarrow}(a(x_n)) > \left(1+\frac{\Delta}{2}\right)\overline{F}^{\leftarrow}(b(x_n))$$

for large enough n. By the definition of $\overline{F}^{\leftarrow}$, it follows that

$$a(x_n) < \overline{F}\left(\left(1+\frac{\Delta}{2}\right)\overline{F}^{\leftarrow}(b(x_n))\right).$$

This relation together with the case (4) of Lemma 2.2 implies

$$\frac{\overline{F}\left(\overline{F}^{\leftarrow}(b(x_n))\right)}{\overline{F}\left(\left(1+\frac{\Delta}{2}\right)\overline{F}^{\leftarrow}(b(x_n))\right)} \leqslant \frac{b(x_n)}{a(x_n)} \underset{n\to\infty}{\to} 1$$

since $a(x) \sim b(x)$ as $x \to \infty$. Thus,

$$\lim_{n\to\infty} \frac{\overline{F}\left(\overline{F}^{\leftarrow}(b(x_n))\right)}{\overline{F}\left(\left(1+\frac{\Delta}{2}\right)\overline{F}^{\leftarrow}(b(x_n))\right)} \leqslant 1.$$

Relations $a(x) \sim b(x)$ and $a(x) \to 0$ as $x \to \infty$ imply $b(x) \to 0$ as $x \to \infty$. Consequently,

$$\overline{F}\Big((1+\Delta/2)\overline{F}^{\leftarrow}(b(x_n))\Big) \to \infty$$

as $n \to \infty$. The contradiction to the definition of $F \in \mathscr{P}_{\mathscr{D}}$ (see Definition 2.8) is again obtained. Lemma 5.3 is now proved. $\square$

We conclude the subsection with the following lemma which ensures the asymptotic equivalence of the generalised inverses of distributions, provided that they belong to $\mathscr{C} \cap \mathscr{P}_{\mathscr{D}}$ and their tails are asymptotically equivalent up to a positive constant.

Lemma 5.4. *Let F and G be two d.f.s such that $F \in \mathscr{C} \cap \mathscr{P}_{\mathscr{D}}$, and $\overline{G}(x) \sim c\overline{F}(x)$ for some positive constant c, as $x \to \infty$. Then*

$$G^{\leftarrow}(1-p) \underset{p\downarrow 0}{\sim} F^{\leftarrow}\Big(1-\frac{p}{c}\Big).$$

Proof. The condition $F \in \mathscr{P}_{\mathscr{D}}$ and Lemmas 5.2, 5.3 imply that

$$x \underset{x\to\infty}{\sim} \overline{F}^{\leftarrow}\big(\overline{F}(x)\big) \underset{x\to\infty}{\sim} \overline{F}^{\leftarrow}\bigg(\frac{\overline{G}(x)}{c}\bigg). \tag{5.14}$$

The conditions $F \in \mathscr{C} \cap \mathscr{P}_{\mathscr{D}}$ and $\overline{G}(x) \sim c\overline{F}(x)$ as $x \to \infty$ imply that $G \in \mathscr{C} \cap \mathscr{P}_{\mathscr{D}}$. Indeed,

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{G}(xy)}{\overline{G}(x)} = \lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{c\overline{F}(xy)}{c\overline{F}(x)} = 1 \qquad \Longrightarrow \qquad G \in \mathscr{C},$$

and for any fixed $y \in (0,1)$,

$$\liminf_{x\to\infty}\frac{\overline{G}(xy)}{\overline{G}(x)} = \liminf_{x\to\infty}\frac{c\overline{F}(xy)}{c\overline{F}(x)} > 1 \qquad \Longrightarrow \qquad G \in \mathscr{P}_{\mathscr{D}}.$$

Hence, $\overline{G}(x) > 0$ for all $x \in \mathbb{R}$, and therefore $\overline{G}^{\leftarrow}(p) \to \infty$ as $p \downarrow 0$. By supposing $x = \overline{G}^{\leftarrow}(p)$ in (5.14), we get

$$\overline{G}^{\leftarrow}(p) \underset{p\downarrow 0}{\sim} \overline{F}^{\leftarrow}\bigg(\frac{1}{c}\,\overline{G}\Big(\overline{G}^{\leftarrow}(p)\Big)\bigg).$$

Due to Lemma 5.1,

$$\overline{G}\Big(\overline{G}^{\leftarrow}(p)\Big) \underset{p\downarrow 0}{\sim} p.$$

Hence, we derive from Lemma 5.3 that

$$\overline{G}^{\leftarrow}(p) \underset{p\downarrow 0}{\sim} \overline{F}^{\leftarrow}\Big(\frac{p}{c}\Big),$$

or, equivalently,

$$G^{\leftarrow}(1-p) \underset{p\downarrow 0}{\sim} F^{\leftarrow}\Big(1-\frac{p}{c}\Big).$$

The lemma is now proved. $\square$

Properties of the special function from the equation (5.3)

Let ξ be a real-valued r.v. such that $\mathbb{E}(\xi^+)^{\varkappa} < \infty$ and $\overline{F}_\xi(x)$ for each $x \in \mathbb{R}$. Let

$$\gamma_\xi(x) = \frac{\Big(\mathbb{E}\big((\xi-x)^+\big)^{\varkappa-1}\Big)^{\varkappa}}{\Big(\mathbb{E}\big((\xi-x)^+\big)^{\varkappa}\Big)^{\varkappa-1}}, \quad x \in \mathbb{R}, \tag{5.15}$$

be a well-defined function from the equation (5.3). In this subsection, we consider the properties of the function γ_ξ and its components. The first such assertion is proved in [108, see Lemma 2.1].

Lemma 5.5. *Let ξ be a real-valued r.v. such that $\mathbb{E}(\xi^+)^{\varkappa} < \infty$ and $\overline{F}_\xi(x) > 0$ for all $x \in \mathbb{R}$. If $\varkappa > 1$, then the function $g(x) := \mathbb{E}\big((\xi-x)^+\big)^{\varkappa}$ is continuously differentiable with*

$$g^{'}(x) = -\varkappa\,\mathbb{E}\big((\xi-x)^+\big)^{\varkappa-1}, \quad x \in \mathbb{R}.$$

If $\varkappa = 1$, then for each $x \in \mathbb{R}$, the derivative from the right is $g^{'}_{+}(x) = -\overline{F}(x)$ and the derivative from the left is $g^{'}_{-}(x) = -\overline{F}(x-0)$.

The following lemmas directly concern the properties of γ_ξ.

Lemma 5.6. *Let ξ be a real-valued r.v. such that $\mathbb{E}(\xi^+)^{\varkappa} < \infty$ and $\overline{F}_\xi(x) > 0$ for all $x \in \mathbb{R}$. Let γ_ξ be the function defined in the equation* (5.15)*. If $\varkappa > 1$, then*

$$\lim_{x\to-\infty} \gamma_\xi(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} \gamma_\xi(x) = 0.$$

Proof. While the main steps of the proof are outlined in [108, See the proof of Theorem 2.1], we provide the complete proof here in more detail.

Since $F_\xi(x) > 0$ for all $x \in \mathbb{R}$, we have

$$\mathbb{E}\big((\xi - x)^+\big)^\varkappa = \varkappa \int_x^\infty (u - x)^{\varkappa-1} \overline{F}_\xi(u) \mathrm{d}u > 0$$

for all $x \in \mathbb{R}$. By the classical Hölder's inequality, we get

$$\begin{aligned}
\gamma_\xi(x) &= \frac{\Big(\mathbb{E}\Big(\big((\xi - x)^+\big)^{\varkappa-1} \mathbb{I}_{(x,\infty)}(\xi)\Big)\Big)^\varkappa}{\Big(\mathbb{E}\big((\xi - x)^+\big)^\varkappa\Big)^{\varkappa-1}} \\
&\leqslant \frac{1}{\Big(\mathbb{E}\big((\xi - x)^+\big)^\varkappa\Big)^{\varkappa-1}} \\
&\qquad \times \left[\Big(\mathbb{E}\big((\xi - x)^+\big)^{(\varkappa-1)\frac{\varkappa}{\varkappa-1}}\Big)^{\frac{\varkappa-1}{\varkappa}} \Big(\mathbb{E}\Big(\mathbb{I}_{(x,\infty)}(\xi)\Big)^\varkappa\Big)^{\frac{1}{\varkappa}}\right]^\varkappa \\
&= \overline{F}_\xi(x),
\end{aligned}$$

which implies the second equality of the lemma.

Let us consider the first equality. It is obvious that

$$\lim_{x \to -\infty} \gamma_\xi(x) = \lim_{x \to \infty} \frac{\left(\mathbb{E}\left(\frac{(\xi + x)^+}{x}\right)^{\varkappa-1}\right)^\varkappa}{\left(\mathbb{E}\left(\frac{(\xi + x)^+}{x}\right)^\varkappa\right)^{\varkappa-1}}. \tag{5.16}$$

The classical c_r-inequality

$$\mathbb{E}\left(|X + Y|^r\right) \leqslant c_r \left(\mathbb{E}|X|^r + \mathbb{E}|Y|^r\right), \qquad c_r = \begin{cases} 1, & \text{if} \quad 0 < r \leqslant 1, \\ 2^{r-1}, & \text{if} \quad r > 1, \end{cases}$$

and the inequality $(a + b)^+ \leqslant a^+ + b^+$, provided that $a, b \in \mathbb{R}$, imply that for a positive x,

$$\begin{aligned}
\mathbb{E}\left(\frac{(\xi + x)^+}{x}\right)^{\varkappa-1} &\leqslant \mathbb{E}\left(\frac{\xi^+ + x}{x}\right)^{\varkappa-1} \\
&\leqslant c_{\varkappa-1}\left(\frac{1}{x^{\varkappa-1}} \mathbb{E}(\xi^+)^{\varkappa-1} + 1\right) < \infty.
\end{aligned}$$

Hence, by the dominated convergence theorem,

$$\lim_{x \to \infty} \left(\mathbb{E}\left(\frac{(\xi + x)^+}{x}\right)^{\varkappa-1}\right)^\varkappa = \left(\mathbb{E} \lim_{x \to \infty} \left(\frac{(\xi + x)^+}{x}\right)^{\varkappa-1}\right)^\varkappa = 1.$$

Similarly,

$$\lim_{x\to\infty}\left(\mathbb{E}\left(\frac{(\xi+x)^+}{x}\right)^{\varkappa}\right)^{\varkappa-1}=1$$

since

$$\mathbb{E}\left(\frac{(\xi+x)^+}{x}\right)^{\varkappa}\leqslant c_{\varkappa}\left(\frac{1}{x^{\varkappa}}\mathbb{E}(\xi^+)^{\varkappa}+1\right)<\infty.$$

Now we can derive the first equality of the lemma by substituting the last two relations into the equality (5.16). The lemma is proved. □

Lemma 5.7. *Let ξ be a real-valued r.v. such that $\mathbb{E}(\xi^+)^{\varkappa}<\infty$ and $\overline{F}_{\xi}(x)>0$ for $x\in\mathbb{R}$. If $\varkappa\geqslant 2$, then the function γ_{ξ} is continuous and strongly decreasing on $\mathbb{R}$.*

Proof. First, consider the case $\varkappa>2$. By denoting $\eta_x=(\xi-x)^+$, we have

$$\gamma_{\xi}(x)=\frac{\left(\mathbb{E}\eta_x^{\varkappa-1}\right)^{\varkappa}}{\left(\mathbb{E}\eta_x^{\varkappa}\right)^{\varkappa-1}}.$$

By Lemma 5.5, we get

$$\gamma_{\xi}^{'}(x)=-\varkappa(\varkappa-1)\,\frac{\left(\mathbb{E}\eta_x^{\varkappa-1}\right)^{\varkappa-1}}{\left(\mathbb{E}\eta_x^{\varkappa}\right)^{\varkappa}}\left(\mathbb{E}\eta_x^{\varkappa-2}\mathbb{E}\eta_x^{\varkappa}-\left(\mathbb{E}\eta_x^{\varkappa-1}\right)^2\right),\ x\in\mathbb{R}.$$

The classical Cauchy-Schwarz inequality and the condition $\overline{F}_{\xi}(x)>0$, $x\in\mathbb{R}$, imply that

$$\mathbb{E}\eta_x^{\varkappa-1}=\mathbb{E}\eta_x^{\frac{\varkappa-2}{2}}\eta_x^{\frac{\varkappa}{2}}<\left(\mathbb{E}\eta_x^{\varkappa-2}\right)^{\frac{1}{2}}\left(\mathbb{E}\eta_x^{\varkappa}\right)^{\frac{1}{2}}.$$

Therefore, $\gamma_{\xi}^{'}(x)<0$ for $x\in\mathbb{R}$, implying the assertion of the lemma in the case $\varkappa>2$.

Now, let us suppose that $\varkappa=2$. In this case,

$$\gamma_{\xi}(x)=\frac{\left(\mathbb{E}\eta_x\right)^2}{\mathbb{E}\eta_x^2}.$$

Using Lemma 5.5, we obtain that the derivative from the right

$$\{\gamma_\xi\}_+'(x) = -2\,\frac{\mathbb{E}\eta_x}{\left(\mathbb{E}\eta_x^2\right)^2}\left(\overline{F}(x)\mathbb{E}\eta_x^2 - \left(\mathbb{E}\eta_x\right)^2\right)$$

is negative for all $x \in \mathbb{R}$, because, by the Hölder inequality,

$$\mathbb{E}\eta_x = \mathbb{E}\,\eta_x \mathbb{I}_{(x,\infty)}(\xi) < \left(\mathbb{E}\,\eta_x^2\right)^{\frac{1}{2}}\left(\mathbb{E}\,\mathbb{I}_{(x,\infty)}(\xi)\right)^{\frac{1}{2}} = \left(\mathbb{E}\,\eta_x^2\right)^{\frac{1}{2}}\left(\overline{F}_\xi(x)\right)^{\frac{1}{2}},$$

where the strictness of the inequality follows from the condition $\overline{F}_\xi(x) > 0$, $x \in \mathbb{R}$. In the same way, we get that the derivative from the left $\{\gamma_\xi\}_-'(x)$ is also negative for all $x \in \mathbb{R}$. Indeed, using Lemma 5.5, we get

$$\{\gamma_\xi\}_-'(x) = -2\,\frac{\mathbb{E}\eta_x}{\left(\mathbb{E}\eta_x^2\right)^2}\left(\overline{F}(x-0)\mathbb{E}\eta_x^2 - \left(\mathbb{E}\eta_x\right)^2\right),$$

which is strictly negative for all $x \in \mathbb{R}$ by the Hölder inequality again, with the observation that

$$\mathbb{E}\eta_x \mathbb{I}_{(x,\infty)}(\xi) = \mathbb{E}\eta_x \mathbb{I}_{[x,\infty)}(\xi), \quad x \in \mathbb{R}.$$

The derived properties of the function γ_ξ imply that this function is continuous and strongly decreasing on $\mathbb{R}$. The lemma is proved. $\square$

Lemma 5.8. *Let ξ be a real-valued r.v. such that $\mathbb{E}(\xi^+)^\varkappa < \infty$ and $F_\xi \in \mathscr{C}$. If $\varkappa > 1$, then*

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\gamma_\xi(xy)}{\gamma_\xi(x)} \leqslant 1.$$

Proof. If $\mathbb{E}(\xi^+)^\varkappa < \infty$, then

$$\mathbb{E}\big((\xi - x)^+\big)^p = p\int_x^\infty (u-x)^{p-1}\overline{F}_\xi(u)\mathrm{d}u$$

for all $0 < p \leqslant \varkappa$. Therefore, for all positive x and y,

$$\begin{aligned}\frac{\gamma_\xi(xy)}{\gamma_\xi(x)} &= \left(\frac{\mathbb{E}\big((\xi - xy)^+\big)^{\varkappa-1}}{\mathbb{E}\big((\xi - x)^+\big)^{\varkappa-1}}\right)^{\varkappa}\left(\frac{\mathbb{E}\big((\xi - x)^+\big)^{\varkappa}}{\mathbb{E}\big((\xi - yx)^+\big)^{\varkappa}}\right)^{\varkappa-1}\\ &= \left(\frac{\int_{xy}^\infty (u-xy)^{\varkappa-2}\overline{F}_\xi(u)\mathrm{d}u}{\int_x^\infty (u-x)^{\varkappa-2}\overline{F}_\xi(u)\mathrm{d}u}\right)^{\varkappa}\left(\frac{\int_x^\infty (u-x)^{\varkappa-1}\overline{F}_\xi(u)\mathrm{d}u}{\int_{xy}^\infty (u-xy)^{\varkappa-1}\overline{F}_\xi(u)\mathrm{d}u}\right)^{\varkappa-1}\end{aligned}$$

$$= \left(\frac{\int_x^\infty (u-x)^{\varkappa-2}\overline{F}_\xi(uy)\mathrm{d}u}{\int_x^\infty (u-x)^{\varkappa-2}\overline{F}_\xi(u)\mathrm{d}u}\right)^{\varkappa} \left(\frac{\int_x^\infty (u-x)^{\varkappa-1}\overline{F}_\xi(u)\mathrm{d}u}{\int_x^\infty (u-x)^{\varkappa-1}\overline{F}_\xi(uy)\mathrm{d}u}\right)^{\varkappa-1}$$

$$\leqslant \left(\sup_{u\geqslant x}\frac{\overline{F}_\xi(uy)}{\overline{F}_\xi(u)}\right)^{\varkappa} \left(\sup_{u\geqslant x}\frac{\overline{F}_\xi(u)}{\overline{F}_\xi(uy)}\right)^{\varkappa-1}. \tag{5.17}$$

If $y \in (0,1)$, then

$$\limsup_{x\to\infty}\frac{\gamma_\xi(xy)}{\gamma_\xi(x)} \leqslant \left(\limsup_{x\to\infty}\sup_{u\geqslant x}\frac{\overline{F}_\xi(uy)}{\overline{F}_\xi(u)}\right)^{\varkappa}.$$

Since $F_\xi \in \mathscr{C}$, the last inequality implies the statement of the lemma. □

Lemma 5.9. *Let ξ be a real-valued r.v. with d.f. F_ξ such that $\mathbb{E}(\xi^+)^{\varkappa} < \infty$ and*

$$\left(\limsup_{x\to\infty}\frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)}\right)^{\varkappa} \left(\limsup_{x\to\infty}\frac{\overline{F}_\xi(x)}{\overline{F}_\xi(xy)}\right)^{\varkappa-1} < 1 \tag{5.18}$$

for some $\varkappa > 1$ and $y > 1$. Then

$$\limsup_{x\to\infty}\frac{\gamma_\xi(xy)}{\gamma_\xi(x)} < 1$$

for some $y > 1$.

Remark 5.1. *Since $\overline{F}_\xi(x) \geqslant \overline{F}_\xi(xv)$ for all $v > 1$, the condition* (5.18) *implies that $F_\xi \in \mathscr{P}_{\mathscr{D}}$. If $F_\xi \in \mathscr{R}_\alpha$ with some $\alpha > 0$, then the condition* (5.18) *holds for all $\varkappa > 1$. If $F_\xi \in \mathscr{ERV}_{\{\alpha,\beta\}} \subset \mathscr{C}$ with some $0 < \alpha < \beta < \infty$, then the condition* (5.18) *holds in the case $1 < \varkappa < \frac{\beta}{\beta-\alpha}$.*

Proof of the lemma. Let $\varkappa > 1$ and $y > 1$ be such that the condition (5.18) holds. By the estimate (5.17), we obtain

$$\limsup_{x\to\infty}\frac{\gamma_\xi(xy)}{\gamma_\xi(x)} \leqslant \limsup_{x\to\infty}\left(\sup_{u\geqslant x}\frac{\overline{F}_\xi(uy)}{\overline{F}_\xi(u)}\right)^{\varkappa} \left(\sup_{u\geqslant x}\frac{\overline{F}_\xi(u)}{\overline{F}_\xi(uy)}\right)^{\varkappa-1}$$
$$= \left(\limsup_{x\to\infty}\frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)}\right)^{\varkappa} \left(\limsup_{x\to\infty}\frac{\overline{F}_\xi(x)}{\overline{F}_\xi(xy)}\right)^{\varkappa-1}.$$

Now the assertion of the lemma follows from the condition (5.18). The lemma is proved. □

Corollary 5.1. *Let $\varkappa \geqslant 2$ and let ξ be a real-valued r.v. with d.f. F_ξ under the condition* (5.18)*. Then the function γ_ξ defined in the equality*

(5.15) *is the t.f. of the d.f.* $1-\gamma_\xi$ *that belongs to the class* $\mathscr{C}\cap\mathscr{P}_{\mathscr{D}}$.

Proof. The assertion of the corollary follows immediately from Lemmas 5.6, 5.7, 5.8 and 5.9. □

Some closure properties

In this subsection, we present some closure properties of d.f.s with respect to the dependence structure and product convolution. The results of this subsection are used to derive Theorem 5.4.

The first lemma given below states the conditions under which the QAI dependence structure between a pair of primary random variables is preserved in a weighted setting.

Lemma 5.10. *Let* ξ_1,ξ_2 *be two real-valued* QAI *r.v.s with d.f.s* $F_{\xi_1}\in\mathscr{D}$, $F_{\xi_2}\in\mathscr{D}$. *Let* θ_1,θ_2 *be two nonnegative nondegenerate at zero r.v.s such that the vectors* (ξ_1,ξ_2) *and* (θ_1,θ_2) *are independent. If* $\max\{\theta_1^p,\theta_2^p\}<\infty$ *for some* $p>\max\{J^+_{F_{\xi_1}},J^+_{F_{\xi_2}}\}$, *then the r.v.s* $\theta_1\xi_1$ *and* $\theta_2\xi_2$ *are* QAI.

The proof of the lemma can be found in [62, Lemma 4].

The following three lemmas, namely, Lemmas 5.11, 5.12 and 5.13, addresses the closure of classes $\mathscr{C}$, $\mathscr{P}_{\mathscr{D}}$ and $\mathscr{D}$ respectively with respect to the product convolution.

Lemma 5.11. *Let* ξ *and* θ *be two independent real-valued r.v.s, and let the d.f.* F_ξ *belong to the class* $\mathscr{C}$. *If* θ *is a nonnegative and nondegenerate at zero r.v. such that* $\mathbb{E}\theta^p<\infty$ *for some* $p>J^+_{F_\xi}$, *then the d.f.* $F_{\theta\xi}$ *of the product* $\theta\xi$ *belongs to the class* $\mathscr{C}$ *as well.*

For the proof of Lemma 5.11, refer to [113, Lemma 2.5].

Lemma 5.12. *Let* ξ *be a real-valued r.v. with d.f.* $F_\xi\in\mathscr{P}_{\mathscr{D}}\cap\mathscr{D}$. *If* θ *is a nonnegative and nondegenerate at zero r.v. that satisfies* $\mathbb{E}\theta^p<\infty$ *for some* $p>J^+_{F_\xi}$ *and is independent of* ξ, *then the product d.f.* $F_{\theta\xi}$ *belongs to the class* $\mathscr{P}_{\mathscr{D}}$.

Proof. Let $y>1$ be fixed under the condition

$$\limsup_{x\to\infty}\frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)}<1,$$

and let $a \in (0,1)$ and $b > 0$ be such that $\mathbb{P}(\theta > b) > 0$. If x is sufficiently large, then

$$\begin{aligned}
\frac{\overline{F}_{\theta\xi}(xy)}{\overline{F}_{\theta\xi}(x)} &= \frac{\mathbb{P}(\theta\xi > xy, \theta \leqslant x^a)}{\mathbb{P}(\theta\xi > x)} + \frac{\mathbb{P}(\theta\xi > xy, \theta > x^a)}{\mathbb{P}(\theta\xi > x)} \\
&\leqslant \frac{\mathbb{P}(\theta\xi > xy, \theta \leqslant x^a)}{\mathbb{P}(\theta\xi > x, \theta \leqslant x^a)} + \frac{\mathbb{P}(\theta > x^a)}{\mathbb{P}(\theta\xi > x, \theta > b)} \\
&\leqslant \frac{\mathbb{P}(\theta\xi > xy, 0 < \theta \leqslant x^a)}{\mathbb{P}(\theta\xi > x, 0 < \theta \leqslant x^a)} + \frac{\mathbb{E}\theta^p}{x^{ap}\mathbb{P}(\theta > b)\mathbb{P}(\xi > x/b)} \\
&\leqslant \sup_{0<u\leqslant x^a} \frac{\overline{F}_\xi\left(\frac{yx}{u}\right)}{\overline{F}_\xi\left(\frac{x}{u}\right)} + \frac{\mathbb{E}\theta^p}{\mathbb{P}(\theta > b)} \frac{1}{x^{ap}\overline{F}_\xi(x)} \frac{\overline{F}_\xi(x)}{\overline{F}\left(\frac{x}{b}\right)} \\
&= \sup_{z\geqslant x^{1-a}} \frac{\overline{F}_\xi(zy)}{\overline{F}_\xi(z)} + \frac{\mathbb{E}\theta^p}{\mathbb{P}(\theta > b)} \frac{1}{x^{ap}\overline{F}_\xi(x)} \frac{\overline{F}_\xi(x)}{\overline{F}\left(\frac{x}{b}\right)}.
\end{aligned} \tag{5.19}$$

Since $F_\xi \in \mathscr{D}$, it is obvious that $\overline{F}_\xi(x) = O\big(\overline{F}_\xi(x/b)\big)$. Moreover,

$$\lim_{x\to\infty} x^q \overline{F}_\xi(x) = \infty \quad \text{for} \quad q > J^+_{F_\xi},$$

see e.g., [106, Lemma 3.5]. If $p > J^+_{F_\xi}$, then there exists $a \in (0,1)$ such that $ap > J^+_{F_\xi}$. For this particular a, the second term on the right side of (5.19) tends to zero if x tends to infinity. Consequently, for this particular $a \in (0,1)$, we have

$$\limsup_{x\to\infty} \frac{\overline{F}_{\theta\xi}(xy)}{\overline{F}_{\theta\xi}(x)} \leqslant \limsup_{x\to\infty} \sup_{z\geqslant x^{1-a}} \frac{\overline{F}_\xi(zy)}{\overline{F}_\xi(z)} = \limsup_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} < 1, \tag{5.20}$$

and thus $F_{\theta\xi} \in \mathscr{P}_{\mathscr{D}}$. The lemma is proved. We only remark that a similar assertion, but for a positive random weight, is proved in [70, Theorem 4.1]. ☐

Lemma 5.13. *Let ξ be a real-valued r.v. with d.f. $F_\xi \in \mathscr{D}$. If θ is a nonnegative and nondegenerate at zero r.v. which is independent of ξ and satisfies $\mathbb{E}\theta^p < \infty$ for some $p > J^+_{F_\xi}$, then for any $y > 0$,*

$$\limsup_{x\to\infty} \frac{\overline{F}_{\theta\xi}(yx)}{\overline{F}_{\theta\xi}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_\xi(yx)}{\overline{F}_\xi(x)} \qquad \text{and}$$

$$\limsup_{x\to\infty} \frac{\overline{F}_{\theta\xi}(x)}{\overline{F}_{\theta\xi}(yx)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_\xi(x)}{\overline{F}_\xi(yx)}.$$

Proof. According to (5.20) we have

$$\limsup_{x\to\infty} \frac{\overline{F}_{\theta\xi}(yx)}{\overline{F}_{\theta\xi}(x)} \leqslant \limsup_{x\to\infty} \sup_{z\geqslant x^{1-a}} \frac{\overline{F}_{\xi}(yz)}{\overline{F}_{\xi}(z)} = \limsup_{x\to\infty} \frac{\overline{F}_{\xi}(yx)}{\overline{F}_{\xi}(x)}$$

for some special $a \in (0,1)$. Hence, the first inequality of the lemma holds. By the same reasoning as in the derivation of (5.19), we obtain

$$\frac{\overline{F}_{\theta\xi}(x)}{\overline{F}_{\theta\xi}(yx)} \leqslant \sup_{z\geqslant x^{1-a}} \frac{\overline{F}(z)}{\overline{F}(yz)} + \frac{\mathbb{E}\theta^p}{\mathbb{P}(\theta > b)} \frac{1}{(yx)^{ap}\overline{F}_{\xi}(x)} \frac{\overline{F}_{\xi}(x)}{\overline{F}_{\xi}(yx/b)}$$

for sufficiently large x, $b > 0$ under the condition $\mathbb{P}(\theta > b) > 0$, and $a \in (0,1)$ such that $ap > J^{+}_{F_\xi}$. The same arguments as in the derivation of (5.20) implies the second inequality of the lemma. The lemma is proved. $\square$

5.3.2 Main proofs

Proof of Theorem 5.3. Let us begin with part (i) of the theorem. By the relation (4.7) of Theorem 4.4, we have

$$\mathbb{E}\Big((S_n^{\xi})^{\beta}\mathbb{I}_{\{S_n^{\xi}>x\}}\Big) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k^{\beta}\,\mathbb{I}_{\{\xi_k>x\}}\Big) \tag{5.21}$$

for all $\beta \in$ [0,1]. In the particular case when $\beta = 0$, we get

$$\overline{F}_{S_n^{\xi}}(x) = \mathbb{P}(S_n^{\xi} > x) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}_n} \overline{F}_{\xi_k}(x) \underset{x\to\infty}{\sim} \overline{H}_n(x). \tag{5.22}$$

We observe that $\overline{F}_{\xi_k}(x) > 0$ for any $k \in \mathcal{I}_n$ if x is sufficiently large. Hence, according to the min-max inequality (see Lemma 2.4), we get that

$$\min_{k\in\mathcal{I}_n} \frac{\overline{F}_{\xi_k}(xy)}{\overline{F}_{\xi_k}(x)} \leqslant \frac{\overline{H}_n(yx)}{\overline{H}_n(x)} \leqslant \max_{k\in\mathcal{I}_n} \frac{\overline{F}_{\xi_k}(xy)}{\overline{F}_{\xi_k}(x)} \tag{5.23}$$

for all $y \in$ (0,1] and for sufficiently large x. This double inequality shows that the d.f. H_n, together with the d.f. $F_{S_n^{\xi}}$, belongs to the class $\mathscr{C}\cap\mathscr{P}_{\mathscr{D}}$. Therefore, by Lemma 5.4,

$$F^{\leftarrow}_{S_n^{\xi}}(q) \underset{q\uparrow 1}{\sim} H_n^{\leftarrow}(q). \tag{5.24}$$

For a $q \in (0{,}1)$, by the equality (5.1) and the min-max inequality (2.43) we derive that

$$
\begin{aligned}
&\frac{\mathrm{HG}_q\Big(S_n^\xi\Big)}{H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+} \\
&\quad\leqslant \max\left\{\frac{F_{S_n^\xi}^{\leftarrow}(q)}{H_n^{\leftarrow}(q)}, \frac{\mathbb{E}\Big(S_n^\xi - F_{S_n^\xi}^{\leftarrow}(q)\Big)^+}{\mathbb{E}\Big(S_n^\xi - H_n^{\leftarrow}(q)\Big)^+}\ \frac{\mathbb{E}\Big(S_n^\xi - H_n^{\leftarrow}(q)\Big)^+}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+}\right\} \\
&\quad\leqslant \frac{F_{S_n^\xi}^{\leftarrow}(q)}{H_n^{\leftarrow}(q)} \max\left\{1, \frac{\mathbb{E}\Big(S_n^\xi - H_n^{\leftarrow}(q)\Big)^+}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+}\right. \\
&\qquad\qquad\left.\times \sup_{u\geqslant H_n^{\leftarrow}(q)} \frac{\mathbb{P}\Big(S_n^\xi > u\frac{F_{S_n^\xi}^{\leftarrow}(q)}{H_n^{\leftarrow}(q)}\Big)}{\mathbb{P}\big(S_n^\xi > u\big)}\right\}
\end{aligned}
\tag{5.25}
$$

because, by the alternative expectation formula (see Lemma 2.5),

$$
\frac{\mathbb{E}(\eta - x_1)^+}{\mathbb{E}(\eta - x_2)^+} = \frac{\int\limits_{x_1}^{\infty}\mathbb{P}(\eta > u)\mathrm{d}u}{\int\limits_{x_2}^{\infty}\mathbb{P}(\eta > u)\mathrm{d}u} \leqslant \frac{x_1}{x_2}\sup_{u\geqslant x_2}\frac{\mathbb{P}(\eta > ux_1/x_2)}{\mathbb{P}(\eta > u)}
$$

for any positive x_1, x_2 and any r.v. η satisfying $\mathbb{P}(\eta > x) > 0$, $x \in \mathbb{R}$. Let us suppose now that $\varepsilon \in (0, 1/2)$. By the relations (5.24) and (5.25) we get that

$$
\begin{aligned}
&\frac{\mathrm{HG}_q\Big(S_n^\xi\Big)}{H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+} \\
&\quad\leqslant (1+\varepsilon)\max\left\{1, \frac{\mathbb{E}\Big(S_n^\xi - H_n^{\leftarrow}(q)\Big)^+}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+}\ \sup_{u\geqslant H_n^{\leftarrow}(q)}\frac{\overline{F}_{S_n^\xi}(u(1-\varepsilon))}{\overline{F}_{S_n^\xi}(u)}\right\}
\end{aligned}
$$

for all q sufficiently close to the unit from the left. Since $H_n^{\leftarrow}(q) \to \infty$ if $q \uparrow 1$, we get from the last estimate that

$$\begin{aligned}
&\limsup_{q\uparrow 1} \frac{\mathrm{HG}_q\Big(S_n^\xi\Big)}{H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+} \\
&\quad \leqslant (1+\varepsilon)\max\left\{1, \limsup_{q\uparrow 1} \frac{\mathbb{E}\Big(S_n^\xi - H_n^{\leftarrow}(q)\Big)^+}{\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+}\right. \\
&\qquad\qquad \left. \times \limsup_{q\uparrow 1} \sup_{u \geqslant H_n^{\leftarrow}(q)} \frac{\mathbb{P}(S_n^\xi > u(1-\varepsilon))}{\mathbb{P}(S_n^\xi > u)}\right\} \\
&\quad = (1+\varepsilon)\max\left\{1, \limsup_{x\to\infty} \frac{\mathbb{E}(S_n^\xi - x)^+}{\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}(\xi_k - x)^+}\right. \\
&\qquad\qquad \left. \times \limsup_{x\to\infty} \sup_{u\geqslant x} \frac{\mathbb{P}(S_n^\xi > u(1-\varepsilon))}{\mathbb{P}(S_n^\xi > u)}\right\}.
\end{aligned}$$

The fact that $F_{S_n^\xi} \in \mathscr{C}$, the relation (4.8) of Theorem 4.4, and the arbitrariness of $\varepsilon \in (0, 1/2)$ imply that

$$\limsup_{q\uparrow 1} \frac{\mathrm{HG}_q\Big(S_n^\xi\Big)}{H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+} \leqslant 1. \tag{5.26}$$

In a similar way, we can obtain

$$\begin{aligned}
&\liminf_{q\uparrow 1} \frac{\mathrm{HG}_q\Big(S_n^\xi\Big)}{H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+} \\
&\quad \geqslant (1-\varepsilon)\min\left\{1, \liminf_{x\to\infty} \frac{\mathbb{E}(S_n^\xi - x)^+}{\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}(\xi_k - x)^+} \liminf_{x\to\infty} \inf_{u\geqslant x} \frac{\overline{F}_{S_n^\xi}(u(1+\varepsilon))}{\overline{F}_{S_n^\xi}(u)}\right\}
\end{aligned}$$

for any $\varepsilon \in (0, 1/2)$, and, by using the analogous arguments, we can derive that

$$\liminf_{q\uparrow 1} \frac{\mathrm{HG}_q\left(S_n^\xi\right)}{H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\left(\xi_k - H_n^{\leftarrow}(q)\right)^+} \geqslant 1.$$

The last estimate, together with the inequality (5.26), finishes the proof of the first part of Theorem 5.3.

Now suppose $\varkappa \geqslant 2$ and all the conditions of part (ii) of Theorem 5.3 are satisfied. By Theorem 5.1,

$$\mathrm{HG}_q(S_n^\xi) = \widetilde{x} + \left(\frac{\mathbb{E}\left(\left(S_n^\xi - \widetilde{x}\right)^+\right)^\varkappa}{1-q}\right)^{1/\varkappa}, \tag{5.27}$$

where $\widetilde{x} = \widetilde{x}(q)$ is the unique solution of the equation

$$\frac{\left(\mathbb{E}\left(\left(S_n^\xi - \widetilde{x}\right)^+\right)^{\varkappa-1}\right)^\varkappa}{\left(\mathbb{E}\left(\left(S_n^\xi - \widetilde{x}\right)^+\right)^{\varkappa}\right)^{\varkappa-1}} = \gamma_{S_n^\xi}(\widetilde{x}) = 1-q.$$

Let H_n be the d.f. defined in part (i) of Theorem 5.3. By the asymptotic relation (5.22), the upper bound in (5.23), and the conditions of the theorem, we have that the d.f. $F_{S_n^\xi}$, together with the d.f. H_n, belongs to the class $\mathscr{C}$. In addition, according to the relation (5.22), the min-max inequality (2.43), and the condition (5.4) we get that

$$\begin{aligned}
&\left(\limsup_{x\to\infty}\frac{\overline{F}_{S_n^\xi}(yx)}{\overline{F}_{S_n^\xi}(x)}\right)^\varkappa \left(\limsup_{x\to\infty}\frac{\overline{F}_{S_n^\xi}(x)}{\overline{F}_{S_n^\xi}(yx)}\right)^{\varkappa-1}\\
&\quad= \left(\limsup_{x\to\infty}\frac{\overline{H}_n(yx)}{\overline{H}_n(x)}\right)^\varkappa \left(\limsup_{x\to\infty}\frac{\overline{H}_n(x)}{\overline{H}_n(yx)}\right)^{\varkappa-1}\\
&\quad\leqslant \left(\max_{k\in\mathcal{I}_n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(yx)}{\overline{F}_{\xi_k}(x)}\right)^\varkappa \left(\max_{k\in\mathcal{I}_n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_k}(yx)}\right)^{\varkappa-1} < 1
\end{aligned} \tag{5.28}$$

for some $y > 1$. Therefore, Corollary 5.1 implies that the d.f. $1 - \gamma_{S_n^\xi}$ belongs to the class $\mathscr{C} \cap \mathscr{P}_{\mathscr{D}}$.

For sufficiently large x, by the alternative expectation formula (see Lemma 2.5), we have that

$$\frac{\left(\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\big((\xi_k - x)^+\big)^{\varkappa-1}\right)^{\varkappa}}{\left(\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\big((\xi_k - x)^+\big)^{\varkappa}\right)^{\varkappa-1}} = \frac{\left((\varkappa-1)\int\limits_x^\infty (u-x)^{\varkappa-2}\overline{H}_n(u)\mathrm{d}u\right)^{\varkappa}}{\left(\varkappa\int\limits_x^\infty (u-x)^{\varkappa-1}\overline{H}_n(u)\mathrm{d}u\right)^{\varkappa-1}}$$

$$= \frac{\left(\mathbb{E}\big((\zeta_n - x)^+\big)^{\varkappa-1}\right)^{\varkappa}}{\left(\mathbb{E}\big((\zeta_n - x)^+\big)^{\varkappa}\right)^{\varkappa-1}} := \gamma_{\zeta_n}(x),$$

where ζ_n is a r.v. with d.f. H_n. By the estimate (5.28), the d.f. $F_{\zeta_n} = H_n$ satisfies the condition (5.18) that, according to Corollary 5.1, implies that the d.f. $1-\gamma_{\zeta_n}$, together with the d.f. $1-\gamma_{S_n^\xi}$, belongs to the class $\mathscr{C}\cap\mathscr{P}_{\mathscr{D}}$.

On the other hand, the relation (4.8) of Theorem 4.4 implies that

$$\gamma_{S_n^\xi}(x) \underset{x\to\infty}{\sim} \gamma_{\zeta_n}(x).$$

Consequently, all the conditions of Lemma 5.4 are satisfied with $c=1$ and the d.f.s $1-\gamma_{S_n^\xi}$, $1-\gamma_{\zeta_n}$. According to this lemma, we get that

$$\widetilde{x}(q) \underset{q\uparrow 1}{\sim} \widehat{x}(q). \tag{5.29}$$

Now we continue the proof of this part in the same way as the proof of part (i). For any $q\in(0,1)$, by the relation (5.27) and the min-max inequality (2.43), we get

$$\frac{\mathrm{HG}_q(S_n^\xi)}{\widehat{x}(q) + \left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa}}$$

$$\leqslant \max\left\{\frac{\widetilde{x}(q)}{\widehat{x}(q)}, \left(\frac{\mathbb{E}\Big(\big(S_n^\xi - \widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}(q)\big)^+\Big)^{\varkappa}}\right)^{1/\varkappa}\right\}$$

$$= \max\left\{\frac{\widetilde{x}(q)}{\widehat{x}(q)}, \left(\frac{\mathbb{E}\Big(\big(S_n^\xi - \widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widetilde{x}(q)\big)^+\Big)^{\varkappa}}\right.\right.$$

$$\left.\left.\times\frac{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}}\right)^{1/\varkappa}\right\}.$$

If q is sufficiently close to the unit from the left, then

$$\frac{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}}=\frac{\int\limits_{\widetilde{x}(q)}^{\infty}(u-\widetilde{x}(q))^{\varkappa-1}\overline{H}_n(u)\mathrm{d}u}{\int\limits_{\widehat{x}(q)}^{\infty}(u-\widehat{x}(q))^{\varkappa-1}\overline{H}_n(u)\mathrm{d}u}$$

$$=\left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}\right)^{\varkappa}\frac{\int\limits_{\widehat{x}(q)}^{\infty}(u-\widehat{x}(q))^{\varkappa-1}\overline{H}_n\left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}u\right)\mathrm{d}u}{\int\limits_{\widehat{x}(q)}^{\infty}(u-\widehat{x}(q))^{\varkappa-1}\overline{H}_n(u)\mathrm{d}u}$$

$$\leqslant\left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}\right)^{\varkappa}\sup_{u\geqslant\widehat{x}(q)}\frac{\overline{H}_n\left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}u\right)}{\overline{H}_n(u)}.$$

Consequently, for q sufficiently close to the unit from the left, we have

$$\frac{\mathrm{HG}_q(S_n^{\xi})}{\widehat{x}(q)+\left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa}}$$

$$\leqslant\frac{\widetilde{x}(q)}{\widehat{x}(q)}\max\left\{1,\left(\frac{\mathbb{E}\Big(\big(S_n^{\xi}-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}\sup_{u\geqslant\widehat{x}(q)}\frac{\overline{H}_n\left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}u\right)}{\overline{H}_n(u)}\right)^{1/\varkappa}\right\}.$$

Since $\widetilde{x}(q)\to\infty$, $\widetilde{x}(q)\sim\widehat{x}(q)$ as $q\uparrow 1$ and the d.f. H_n belongs to the class $\mathscr{C}$, the last estimate and the asymptotic relation (4.8) of Theorem 4.4 imply that

$$\limsup_{q\uparrow 1}\frac{\mathrm{HG}_q(S_n^{\xi})}{\widehat{x}(q)+\left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa}}\leqslant 1.$$

In a similar way, we can derive that

$$\liminf_{q\uparrow 1} \frac{\mathrm{HG}_q(S_n^\xi)}{\widehat{x}(q) + \left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}(q)\big)^+\Big)^\varkappa\right)^{1/\varkappa}} \geqslant 1.$$

This finishes the proof of the second part of the theorem. □

Proof of Theorem 5.4. We begin with the case $\varkappa = 1$. From Lemma 5.10 and the hypotheses of the theorem, the collection of r.v.s $\{\theta_1\xi_1, \ldots, \theta_n\xi_n\}$ follows the pQAI dependence structure. By Lemma 5.11, we have that the d.f. $F_{\theta_k\xi_k}$ belongs to the class $\mathscr{C}$ for any index $k \in \{1, 2, \ldots, n\}$. For any $k \in \{1, \ldots, n\}$, we have $\mathbb{E}\theta_k^p < \infty$ given that $J^+_{F_{\xi_k}} < p$. By [106, Lemma 3.5 (iii)] we also have

$$\sup\left\{v : \int_{[0,\infty)} x^v \mathrm{d}F_{\xi_k}(x) < \infty\right\} \leqslant J^+_{F_{\xi_k}}$$

since $F_{\xi_k} \in \mathscr{D}$. As $\mathbb{E}\xi_k^+ < \infty$, it follows that

$$1 \leqslant \sup\left\{v : \mathbb{E}(\xi_k^+)^v < \infty\right\} \leqslant J^+_{F_{\xi_k}} < p,$$

and therefore $\mathbb{E}\theta_k < \infty$. This implies that $\mathbb{E}\theta_k\xi_k^+ < \infty$. Finally, by Lemma 5.12, we derive that the d.f. $F_{\theta_k\xi_k} \in \mathscr{P}_\mathscr{D}$ for any $k \in \{1, 2, \ldots, n\}$. Hence, the collection $\{\theta_1\xi_1, \ldots, \theta_n\xi_n\}$ satisfies all the conditions of part (i) of Theorem 5.3. This implies the statement of Theorem 5.4 (i).

Now let us consider part (ii) of the theorem. If $\varkappa \geqslant 2$, then Theorem 5.4 (ii) follows from Theorem 5.3 (ii) by completely analogous reasoning to that in the first part of the proof, except that Lemma 5.13 is used instead of Lemma 5.12. □

5.4 Examples

In this section, we present an example showing how Theorem 5.3 can be applied to the evaluation of the HG risk measure when the confidence level is close to unity and the leading risk has a consistently varying distribution.

Example 5.1. *Let us consider a collection of r.v.s* $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ *with* *the following d.f.s:*

$$F_1(x) = \left(1 - \frac{1}{(x+1)^3}\right) \mathbb{I}_{[0,\infty)}(x),$$
$$F_2(x) = \left(1 - \frac{1}{2(x+1)^3}\right) \mathbb{I}_{[0,\infty)}(x),$$
$$F_3(x) = F_4(x) = (1 - \mathrm{e}^{-x})\, \mathbb{I}_{[0,\infty)}(x).$$

As in the previous examples, we analyse both the case of independence and the case of dependence. In the first case, the r.v.s are mutually independent, while in the second one, the dependence structure is defined as follows:

- *The random vector (ξ_1, ξ_3) follows the dependence structure implied by the Frank copula with parameter $\vartheta = 5$:*

$$\mathbb{P}(\xi_1 \leqslant x_1, \xi_3 \leqslant x_3) = -\frac{1}{5} \log\left(1 + \frac{(\mathrm{e}^{-5F_1(x_1)} - 1)(\mathrm{e}^{-5F_3(x_3)} - 1)}{\mathrm{e}^{-5} - 1}\right);$$

- *The dependence of (ξ_2, ξ_4) is controlled by the Frank copula with parameter $\vartheta = 10$:*

$$\mathbb{P}(\xi_2 \leqslant x_2, \xi_4 \leqslant x_4) = -\frac{1}{10} \log\left(1 + \frac{(\mathrm{e}^{-10F_2(x_1)} - 1)(\mathrm{e}^{-10F_4(x_3)} - 1)}{\mathrm{e}^{-10} - 1}\right);$$

- *The pairs $\{\xi_1, \xi_2\}$, $\{\xi_1, \xi_4\}$, $\{\xi_2, \xi_3\}$, $\{\xi_3, \xi_4\}$ each consist of independent r.v.s.*

We derive the asymptotic formulas for the HG_q *risk measure in the cases of the power Young functions $\varphi(t) = t^{\varkappa}$ with $\varkappa = 1$ and $\varkappa = 2$.*

It is easy to see that the collection of r.v.s satisfies the conditions of Theorem 5.3 with $\mathcal{I}_4 = \{1, 2\}$. As it was shown in Example 2.2, the exponential distributions F_3 and F_4 are light-tailed and, as a result, not consistently varying. On the other hand, F_1 is a regularly varying Pareto distribution (see Example 2.3), while $F_2 \in \mathscr{R}$ since

$$\lim_{x\to\infty} \frac{\overline{F}_2(xy)}{\overline{F}_2(x)} = \lim_{x\to\infty} \left(\frac{x+1}{xy+1}\right)^3 = y^{-3}.$$

To show that the exponential tails are asymptotically dominated by the tail $\overline{F}_1$, observe that

$$\lim_{x\to\infty}\frac{\overline{F}_3(x)}{\overline{F}_1(x)}=\lim_{x\to\infty}\frac{(x+1)^3}{\mathrm{e}^x}=0.$$

The condition (5.4) holds as well for $\varkappa=2$ since for all $y>1$ we have

$$\left(\max_{k\in\mathcal{I}_4}\limsup_{x\to\infty}\frac{\overline{F}_k(yx)}{\overline{F}_k(x)}\right)^{\varkappa}\left(\max_{k\in\mathcal{I}_4}\limsup_{x\to\infty}\frac{\overline{F}_k(x)}{\overline{F}_k(yx)}\right)^{\varkappa-1}$$
$$=\left(\limsup_{x\to\infty}\left(\frac{x+1}{xy+1}\right)^3\right)^2\limsup_{x\to\infty}\left(\frac{xy+1}{x+1}\right)^3=y^{-3}<1.$$

- In the case $\varkappa=1$, we have

$$\mathrm{HG}_q(S_4^{\xi})\underset{q\uparrow 1}{\sim}H_4^{\leftarrow}(q)+\frac{1}{1-q}\left(\mathbb{E}(\xi_1-H_4^{\leftarrow}(q))+\mathbb{E}(\xi_2-H_4^{\leftarrow}(q))\right),$$

where $H_4^{\leftarrow}(q)$ is the quantile function of the d.f.

$$H_4=\max\left\{0,\,1-\overline{F}_1-\overline{F}_2\right\}.$$

For sufficiently large x,

$$H_4(x)=1-\frac{3}{2}\frac{1}{(1+x)^3}.$$

Therefore,

$$H_4^{\leftarrow}(q)=\frac{1}{\sqrt[3]{\frac{2}{3}(1-q)}}-1$$

for q sufficiently close to unity. For large enough x,

$$\mathbb{E}(\xi_1-x)^+=\frac{1}{2(x+1)^2},\quad \mathbb{E}(\xi_2-x)^+=\frac{1}{4(x+1)^2}.$$

Therefore, in the case $\varkappa=1$,

$$\mathrm{HG}_q(S_4^{\xi})\underset{q\uparrow 1}{\sim}\frac{1}{\sqrt[3]{\frac{2}{3}(1-q)}}-1+\frac{1}{1-q}\,\frac{3}{4}\left(\sqrt[3]{\frac{2}{3}(1-q)}\right)^2$$
$$=\frac{1}{\sqrt[3]{1-q}}\left(\frac{3}{2}\right)^{\frac{4}{3}}-1.$$

The graph of this function, together with the values of $\mathrm{HG}_q(S_4^\xi)$ obtained using the MC method, is depicted in Figure 5.1. To

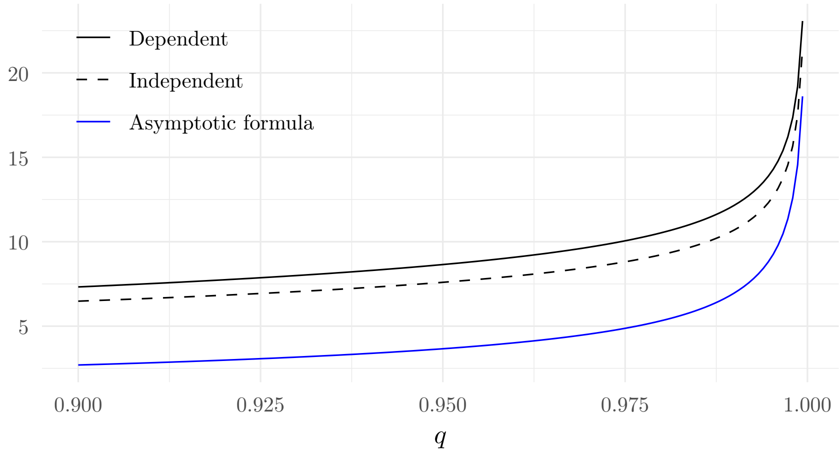

Figure 5.1: Simulated and asymptotically approximated values of $\mathrm{HG}_q(S_4^\xi)$ in the case $\varkappa = 1$.

better reflect the convergence of the asymptotic formula, we also present the previous graph in terms of relative absolute errors of the asymptotic formula (see Figure 5.2).

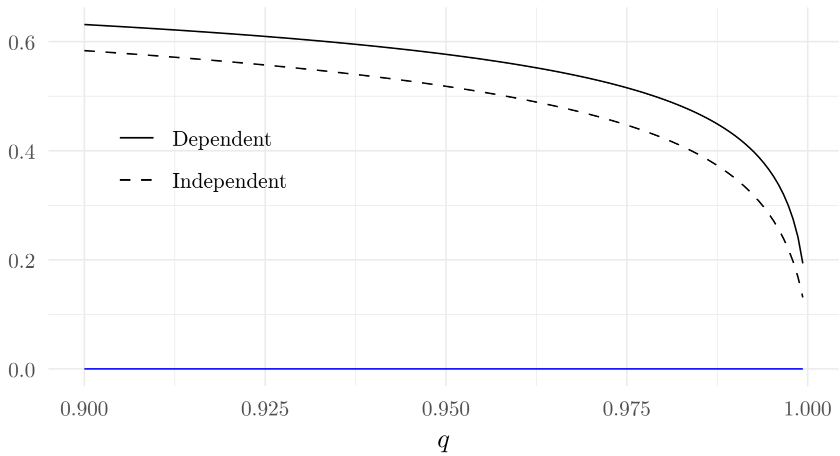

Figure 5.2: Relative absolute errors of the asymptotic approximation of $\mathrm{HG}_q(S_4^\xi)$ in the case $\varkappa = 1$.

- If $\varkappa = 2$, then

$$\mathrm{HG}_q(S_4^\xi) \underset{q\uparrow 1}{\sim} \widehat{x}(q) + \frac{1}{\sqrt{1-q}}\left(\mathbb{E}\big((\xi_1 - \widehat{x}(q))^+\big)^2 + \mathbb{E}\big((\xi_2 - \widehat{x}(q))^+\big)^2\right)^{1/2},$$

where $\widehat{x}(q)$ is the solution of the equation

$$\frac{\mathbb{E}(\xi_1 - x)^+ + \mathbb{E}(\xi_2 - x)^+}{\mathbb{E}\big((\xi_1 - x)^+\big)^2 + \mathbb{E}\big((\xi_2 - x)^+\big)^2} = 1 - q.$$

In the case under consideration,

$$\begin{aligned} \mathbb{E}(\xi_1 - x)^+ &= \frac{1}{2(x+1)^2}, \quad & \mathbb{E}(\xi_2 - x)^+ &= \frac{1}{4(x+1)^2}, \\ \mathbb{E}\big((\xi_1 - x^+\big)^2 &= \frac{1}{x+1}, \quad & \mathbb{E}\big((\xi_2 - x^+\big)^2 &= \frac{1}{2(x+1)}. \end{aligned}$$

Hence,

$$\widehat{x}(q) = \sqrt[3]{\frac{3}{8(1-q)}} - 1$$

and

$$\mathrm{HG}_q(S_4^\xi) \underset{q\uparrow 1}{\sim} \frac{3}{2}\sqrt[3]{\frac{3}{1-q}} - 1.$$

The graph of this function, together with the values of $\mathrm{HG}_q(S_4^\xi)$ obtained using the MC method, can be seen in Figure 5.3. To

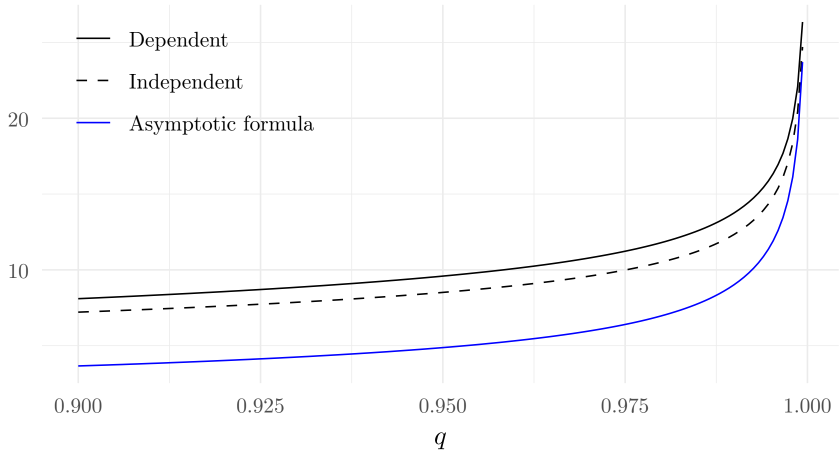

Figure 5.3: Simulated and asymptotically approximated values of $\mathrm{HG}_q(S_4^\xi)$ in the case $\varkappa = 2$.

better illustrate the convergence of asymptotic formula we again present the graph of relative absolute errors (see Figure 5.4).

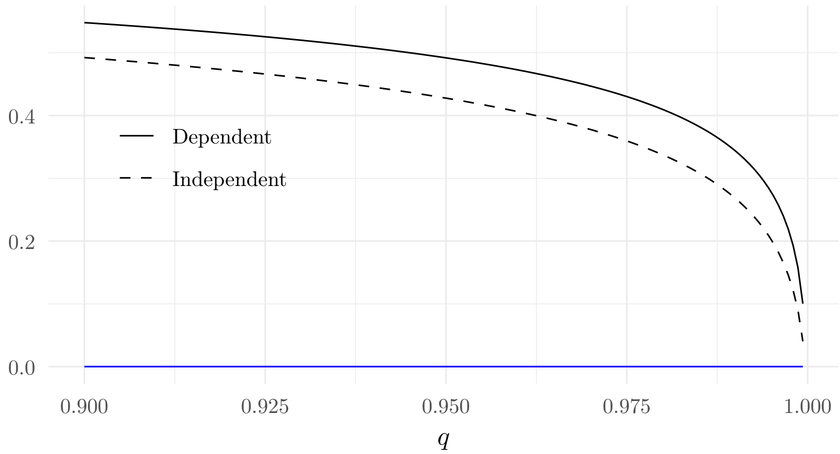

Figure 5.4: Relative absolute errors of asymptotic approximation of $\mathrm{HG}_q(S_4^{\xi})$ in the case $\varkappa = 2$.

6 Conclusions

In this dissertation, we investigated the asymptotic properties of three types of transformations of heavy-tailed distributions. Specifically, we considered the product distributions of zero-mean light-tailed normal random variables, which exhibit heavy-tailed behaviour when at least three multipliers are involved, the generalised tail moments of random sums with heavy tailed summands transformed by a specific function, and the Haezendonck-Goovaerts risk measure for sums of heavy-tailed random risks.

In the first part of the dissertation, we investigated the asymptotic tail behaviour of the product distributions of independent zero-mean normal r.v.s. The main contributions are stated in Theorem 3.2 and Corollary 3.2, which provide more precise asymptotic estimates for the products of standard normal and zero-mean normal r.v.s with arbitrary variances, respectively. Compared to the exact asymptotic equivalence obtained in [6] (see Theorem 3.1 and Proposition 3.1), we provide the asymptotic formulas with the first order remaining term, thus offering more insight into the decay rate of the tail probability.

In the second part of the dissertation, we obtained the asymptotic formulas of the generalised moments for the random(-ly weighted) sums with possibly dependent dominatedly varying primary r.v.s under the transformation satisfying certain eventual properties. The main findings stated in Theorems 4.5 and 4.6 provide asymptotic bounds given by the sums of generalised tail moments of transformed individual summands with the correcting constants defined via L-indices. Compared with the previous results in the literature, our findings improve and generalise them in several ways. Firstly, by placing the L-indices of individual summands inside the bounding sums, we obtain sharper asymptotic bounds under the pQAI dependence structure. Secondly, rather than focusing on a family of power functions with nonnegative integer exponents, we consider a broader class of transformations. Finally, in the case of randomly weighted sums, we substitute the boundedness requirement on random weights with a less restrictive moments condition.

In the third part, we explored the asymptotic properties of the HG risk measure for the random(-ly weighted) sums with summands from the class of consistently varying distributions under the same pQAI dependence structure. The core results given in Theorems 5.3 and 5.4 enable to asymptotically express the risk measure for the sum of random risks as a certain functional, structurally resembling the one obtained by Tang and Yang in [108] (see Theorem 5.1), dependent on the marginal distributions of individual risks. To obtain the asymptotic estimate, there is no need to evaluate the complex distribution of the sum of r.v.s. Moreover, Theorem 5.3, formulated for the unweighted primary r.v.s, implies that to find the asymptotics of the HG risk measure for the sum S_n^{ξ} it is sufficient to determine the distributions of the asymptotically dominant summands.

To illustrate and empirically validate the obtained theoretical results, in all three parts of the dissertation, we performed MC simulation studies. We constructed specific examples of products of (standard) normal r.v.s and random(-ly weighted) sums with pQAI dependence structure implied by selected copulas. The experiments verified the accuracy of the derived asymptotic formulas presented in Chapters 3, 4 and 5.

Lastly, we briefly discuss possible directions for future research. First, building on the asymptotic results regarding the behaviour of the tail probability of products of normal r.v.s presented in Chapter 3, future studies could extend the results for the product convolutions of not necessarily zero-mean normal distributions. What is more, additional error terms beside the leading-order one in the expansion could potentially be uncovered, resulting in more accurate asymptotic approximations.

Second, the results of this dissertation in Chapters 4, 5 could be extended to the infinite sums

$$S_{\infty}^{\theta\xi} := \sum_{k=1}^{\infty} \theta_k \xi_k \qquad \text{and} \qquad S_{\infty}^{\xi} := \sum_{k=1}^{\infty} \xi_k,$$

considered in [25, 132] among others, or for the maximums of the summands

$$M_n^{\theta\xi} := \max\{\theta_1\xi_1, \ldots, \theta_n\xi_n\}, \qquad M_n^{\xi} := \max\{\xi_1, \ldots, \xi_n\} \tag{6.1}$$

and

$$M_\infty^{\theta\xi} := \sup_{1\leqslant k<\infty}\{\theta_k\xi_k\}, \qquad M_\infty^{\xi} := \sup_{1\leqslant k<\infty}\{\xi_k\}, \tag{6.2}$$

as in [107, 114, 132]. Here, (6.1) and (6.2) admit a direct interpretation in the discrete-time risk model, corresponding to the finite- and ultimate-time ruin probabilities (see, for instance, [71, Chapter 1]) respectively:

$$\psi(x,n) := \mathbb{P}\left(S_k^{\theta\xi} > x, \quad \text{for some} \quad 1 \leqslant k \leqslant n\right) = \mathbb{P}\left(M_n^{\theta\xi} > x\right) \qquad \text{and}$$
$$\psi(x) := \mathbb{P}\left(S_k^{\theta\xi} > x, \quad \text{for some} \quad 1 \leqslant k < \infty\right) = \mathbb{P}\left(M_\infty^{\theta\xi} > x\right).$$

In this context, x is the initial capital of an insurance company at the present time and $S_k^{\theta\xi}$ corresponds to the present value of the total net loss accumulated during the first k periods. What is more, regarding the asymptotics of the generalised tail moments of random(-ly weighted) sums one could consider the relations (4.9) and (4.11) under a possibly wider class of transformations and/or a different dependence structure between the primary r.v.s.

Third, it is important to note that the results concerning the asymptotics of the HG risk measure under the power Young function in Theorems 5.3 and 5.4 do not cover the case where the exponent $\varkappa \in (1,2)$. Moreover, similar results could potentially be obtained for the wider class $\mathscr{D}$; however, the asymptotic formulas will likely require additional constants, resulting in less strict asymptotic bounds, or additional constraints should be imposed to preserve exact asymptotic relations. Furthermore, one could consider a wider class of Young functions, similarly as in [109]. Or, possibly, further restrict the class of transformations considered in Theorems 4.5 and 4.6 by taking the intersection with the class of normalised Young functions.

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7 Santrauka (Summary in Lithuanian)

7.1 Įvadas

Skirstinio sunkiauodegiškumo savybė, kuri nusako skirstinio uodegos funkcijos (u.f.) elgesį, intuityviai reiškia, jog tokia u.f. gęsta reikšmingai greičiau nei bet kurio eksponentinio skirstinio u.f. Priešingai nei klasikinėje tikimybių teorijoje ir statistikoje, kurioje didelis dėmesys skiriamas centrinei ribinei teoremai, skirstinių normalumui ir centriniams momentams, sunkiauodegio skirstinio elgesį nusako didelės reikšmės uodegose [92], o asimptotinis uodegos elgesys yra kertinė jo charakteristika.

Bėgant metams sunkiauodegiai skirstiniai sulaukė didelio dėmesio skirtingose disciplinose, tokiose kaip tikimybių teorija, ekonomika ir finansai, draudimas ar aplinkotyra, ir yra dažnai pasitelkiami modeliuoti santykinai retus, bet ekstremalius įvykius, kuriuos klasikiniai skirstiniai yra linkę nuvertinti. Dėl sistemų, kuriose sunkiauodegiai skirstiniai yra taikomi, kompleksiškumo, dažnai yra nagrinėjamos įvairios tokiais skirstiniais apibrėžtų atsitiktinių dydžių (a.d.s) transformacijos, pavyzdžiui: sumos ([56, 25, 132, 26]), sandaugos ([30, 31, 68], [74, 5 skyrius]), ar tam tikri funkcionalai, kaip apibendrinti momentai (žr. [76, 105]) ir rizikos matai (žr. [108, 80, 109]).

Šioje disertacijoje nagrinėjamos trys specifinės sunkiauodegių skirstinių transformacijos ir jų asimptotika. Tyrimo problematika ir tikslai trumpai apžvelgiami trijuose toliau sekančiuose skyreliuose.

7.1.1 Normaliųjų atsitiktinių dydžių sandaugos: nuo lengvų iki sunkių uodegų

Normalusis (Gauso) skirstinys dažnai yra pateikiamas kaip lengvą uodegą turinčio skirstinio pavyzdys. Jis yra vienas dažniausiai tikimybių teorijoje ir statistikoje sutinkamų skirstinių, kurio savybės yra plačiai išnagrinėtos ir gerai žinomos. Kita vertus, taip pasiskirsčiusių a.d.s sandaugos skirstinio savybės iki šiol yra plačiai nagrinėjamos mokslinėje literatūroje ([82, 34, 54, 55]). Dalyje straipsnių yra gaunamos tikslios dviejų normaliųjų a.d.s sandaugos skirstinio tankio išraiškos [123, 133, 33, 82, 34, 54, 55]. Visgi, gautos formulės nėra uždaro pavidalo ir yra apibrėžiamos per integralus ir funkcijų eilutes – tokias išraiškas sudėtinga pritaikyti praktiniuose skaičiavimuose ir apibendrinti atvejams, kai sandauga sudaryta iš trijų ir daugiau daugiklių. Dėl to vertinant normaliųjų a.d.s sandaugos skirstinių elgesį didelę reikšmę turi asimptotinės aproksimacijos. Jų svarba dar labiau išryškėja analizuojant tokias sandaugas sunkiauodegių skirstinių kontekste – galima parodyti, kad trijų ir daugiau nulinio vidurkio daugiklių atveju, normaliųjų a.d.s sandauga yra sunkiauodegė (žr. 7.1 lemą). Formaliai, jei $\xi_1, \dots, \xi_n$, $n \in \mathbb{N}$, yra nepriklausomi standartiniai normalieji a.d.s, kurių sandauga žymima

$$\Pi_n := \prod_{k=1}^{n} \xi_n,$$

tuomet Π_n yra lengvauodegė, kai $n < 3$, ir turi sunkią uodegą, kai $n \geqslant 3$.

Palyginti neseniai [6] straipsnyje buvo gautos asimptotinės išraiškos, kurias galima taikyti skirstinio uodegos funkcijai $\mathbb{P}(\Pi_n > x)$, t. y. galima parodyti, jog galioja

$$\mathbb{P}\left(\Pi_n > x\right) \underset{x\to\infty}{\sim} W_n(x).$$

Čia W_n yra realioji funkcija. Disertacijoje siekiama patikslinti šią asimptotinę išraišką papildomai įtraukiant liekamuosius(-ąjį) narius(-į), kurie suteiktų daugiau informacijos apie asimptotinės aproksimacijos tikslumą.

7.1.2 Sunkiauodegių atsitiktinių dydžių (svorinės) sumos

Atsitiktinės baigtys modeliuojamoje sistemoje dažnai gali būti išreikštos kaip tam tikrų (sunkiauodegių) a.d.s suma. Pavyzdžiui, tai gali pasitaikyti draudimo kontekste, kai vertinama bendra polisų portfelio draudimo išmokų suma, arba finansų srityje, kuomet agreguoti pozicijų portfelio nuostoliai gali būti išreikšti kaip atskirų pozicijų nuostolių suma. Atvejais, kai kiekvieno individualaus komponento indėlis sistemoje yra stochastiškas, minėtos sumos gali būti apibendrintos kiekvieną sumos dėmenį padauginus iš atsitiktinio svorio.

Formaliai, tegu $n \in \mathbb{N}$, $\{\xi_1, \ldots, \xi_n\}$ yra realiųjų, galimai priklausomų sunkiauodegių a.d.s, dar vadinamų *pagrindiniais*, rinkinys, o $\{\theta_1, \ldots, \theta_n\}$ – neneigiamų, neišsigimusių nulyje a.d.s, dar vadinamų *atsitiktiniais svoriais*, rinkinys. Pagrindinių a.d.s suma bus žymima

$$S_n^{\xi} := \sum_{k=1}^{n} \xi_k = \xi_1 + \ldots + \xi_n, \tag{7.1}$$

o svorinė a.d.s suma, atitinkamai:

$$S_n^{\theta\xi} := \sum_{k=1}^{n} \theta_k \xi_k = \theta_1 \xi_1 + \ldots + \theta_n \xi_n. \tag{7.2}$$

Pastaraisiais metais sumų (7.1) ir (1.2) asimptotinės savybės buvo plačiai nagrinėjamos tikimybių teorijos literatūroje, ypač rizikos ir ekstremalių reikšmių teorijų kontekstuose. Konkrečiai, daugelyje tyrimų buvo nagrinėjama atsitiktinių sumų S_n^{ξ} ir $S_n^{\theta\xi}$ skirstinio uodegos funkcijų (u.f.s)

$$\mathbb{P}\left(S_n^{\xi} > x\right) \qquad \text{ir} \qquad \mathbb{P}\left(S_n^{\theta\xi} > x\right), \qquad \text{kai} \qquad x \to \infty,$$

asimptotika, išreiškiant minėtas u.f.s per atskirų sumos dėmenų u.f. sumas (žr. pvz.: [25, 56, 26, 59, 62, 76, 107, 113, 114, 132]). Kartu, dalis autorių nagrinėjo atsitiktinių (svorinių) sumų uodegos momentus

$$\mathbb{E}\left(\varphi(S_n^{\xi})\, \mathbb{I}_{\{S_n^{\xi} > x\}}\right) \qquad \text{ir} \qquad \mathbb{E}\left(\varphi(S_n^{\theta\xi})\, \mathbb{I}_{\{S_n^{\xi} > x\}}\right), \tag{7.3}$$

kai $x \to \infty$. Čia $\varphi : \mathbb{R} \to \mathbb{R}$ yra tam tikra mati funkcija. Palyginti su uodegos funkcijų elgesio tyrimais, uodegos momentų (7.3) asimptotika literatūroje buvo analizuota rečiau (žr. [62, 76, 128, 105]), dažniausiai, atveju $\varphi(x) = x$.

Šiame darbe toliau nagrinėjami atsitiktinių svorinių sumų $S_n^{\theta\xi}$, su sunkiauodegiais, galimai priklausomais pagrindiniais a.d.s, apibendrintų uodegos momentų asimptotiniai sąryšiai. Siekiama apibrėžti tokią mačių funkcijų klasę $\mathcal{K}$, kad jei $\varphi \in \mathcal{K}$, tuomet transformuotos atsitiktinės sumos uodegos momentą galima asimptotiškai aprėžti transformuotų sumos dėmenų uodegos momentų sumomis, t. y.

$$\mathbb{E}\left(\varphi(S_n^{\theta\xi})\mathbb{I}_{\{S_n^{\theta\xi}>x\}}\right) \underset{x\to\infty}{\gtrsim} \sum_{k=1}^{n} C_{\xi_k}\mathbb{E}(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta\xi>x\}}),$$
$$\mathbb{E}\left(\varphi(S_n^{\theta\xi})\mathbb{I}_{\{S_n^{\theta\xi}>x\}}\right) \underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{C_{\xi_k}}\mathbb{E}(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta\xi>x\}}).$$

Čia C_{ξ_k} yra koreguojanti konstanta, kuri priklauso nuo atsitiktinio dydžio (a.d.) ξ_k.

7.1.3 Agreguotų sunkiauodegių rizikų vertinimas rizikos matais

Finansų ir draudimo veikla yra neatsiejama nuo ekstremalių skirstinio uodegos įvykių vertinimo. Vieni pagrindinių įrankių, leidžiančių įvertinti tokių įvykių rizikingumą, yra rizikos matai – funkcionalai, kurie finansinėms pozicijoms priskiria skaitinę reikšmę, nusakančią jų rizikos lygį. Iš vienos pusės, toks rizikos vertinimas gali būti naudojamas kaip priemonė skirta nustatyti papildomo kapitalo poreikį nepalankiųjų scenarijų atveju. Kita vertus, jis gali būti interpretuojamas kaip draudimo įmokos nustatymo principas draudimo kontekste.

Pamokos išmoktos po finansų krizių bei vis sudėtingėjančios finansų rinkos išryškino kai kurių plačiai naudojamų rizikos matų trūkumus. Pavyzdžiui, gerai žinomas vertės pokyčio rizikos matas (VaR), bendruoju atveju, netenkina diversifikacijos principo ir, dėl savo apibrėžimo, neatsižvelgia į potencialių nuostolių dydį virš pasirinkto procentilinio rėžio.

Minėti iššūkiai paskatino ieškoti kitų, labiau teoriniu pagrindu grįstų alternatyvų. Viena jų – deficito vidurkis (ES), kuris, priešingai nei VaR rizikos matas, pagal Artzner ir bendraautorių [9] terminologiją, yra suderintas rizikos matas. Kitas suderintas rizikos matas, suteikiantis dar daugiau lankstumo atliekant rizikos vertinimą, yra vadinamasis Haezendonck-Goovaerts (HG) rizikos matas [60], kurio elgesys priklauso nuo pasirinktos normalizuotos Young funkcijos φ. Galima įsitikinti, kad atveju, kai φ yra tiesinė funkcija, HG matas yra ekvivalentus ES matui.

Svarbu atkreipti dėmesį, kad dėl savo apibrėžimo, kuris įtraukia kvantilių funkcijas, HG rizikos matą praktiniuose skaičiavimuose gali būti sudėtinga tiesiogiai įvertinti. Štai [108] straipsnyje paminima, jog analitinė HG rizikos mato išraiška bendruoju atveju nėra pasiekiama, nors pasirinkus tam tikras normalizuotas Young funkcijas galima gauti paprastesnes išraiškas (daugiau žr. [108] ir 7.8 teoremą). Dėl šių priežasčių literatūroje nagrinėjami tiek statistiniai HG rizikos mato įvertiniai ir jų savybės (žr. [119, 78, 134] ir kt.), tiek asimptotinės išraiškos, kai pasikliovimo lygmuo q artėja į 1. Šioje disertacijoje, pasirinkus antrąją kryptį, siekiama gauti asimptotines išraiškas atsitiktinėms (svorinėms) sumoms S_n^{ξ} ir $S_n^{\theta\xi}$, atveju, kai normalizuota Young fukcija yra laipsninė, t. y. $\varphi(t) = t^{\varkappa}$. Pasiremiant [108] rezultatais, HG rizikos matą norima asimptotiškai išreikšti funkcionalu, kuris priklausytų nuo atskirų rizikų ξ_k ar $\theta_k\xi_k$, $k = 1, \ldots, n$, ir kurio skaičiavimui nereikėtų vertinti atsitiktinių sumų S_n^{ξ} ir $S_n^{\theta\xi}$ skirstinių.

7.2 Apibrėžimai

7.2.1 Skirstinių klasės

Šiame skyrelyje trumpai apžvelgiamos pagrindinės disertacijoje naudojamos skirstinių klasės ir jų savybės. Bene aktualiausia, vadinamoji sunkiauodegių skirstinių klasė.

Prieš pateikiant jos apibrėžimą, svarbu įvesti skirstinio atramos sąvoką. Pasiskirstymo funkcijos (p.f.) F atrama yra

$$\operatorname{supp}\{F\} := \{x\colon F(x+\delta) - F(x-\delta) > 0 \quad \text{visiems} \quad \delta > 0\}.$$

Sakoma, kad p.f. F turi atramą $\mathbb{R}$, jei $\operatorname{supp}\{F\} \subseteq \mathbb{R}$.

Apibrėžimas 7.1. *Pasiskirstymo funkcija F, turinti atramą $\mathbb{R}$, vadinama* sunkiauodege*, rašoma $F \in \mathscr{H}$, jei kiekvienai $\delta > 0$ teisinga*

$$\int_{\mathbb{R}} \mathrm{e}^{\delta x} \mathrm{d}F(x) = \infty.$$

Kitaip tariant, 7.1 apibrėžimas sako, kad a.d. ξ yra sunkiauodegis, jei jo momentus generuojanti funkcija $M_\xi(\delta) = \mathbb{E}(e^{\delta\xi}) = \infty$ kiekvienai

$\delta > 0$. Sakoma, kad a.d., arba jo p.f., yra *lengvauodegė*, jei ji nėra sunkiauodegė.

Toliau pateikiami pagrindinių klasės $\mathscr{H}$ poklasių, dar vadinamų skirstinių reguliarumo klasėmis, apibrėžimai.

Viena labiausiai išnagrinėtų reguliarumo klasių – vadinamoji reguliariai kintančių skirstinių klasė, pirmiausiai formalizuota Karamata [64].

Apibrėžimas 7.2. *P.f. F, turinti atramą $\mathbb{R}$, vadinama* turinčia reguliariai kintančią uodegą *(arba tiesiog* reguliariai kintančia*) su indeksu $\alpha \geqslant 0$, rašoma $F \in \mathscr{R}_\alpha$, jei bet kuriam $y > 0$ teisinga*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}.$$

Klasė $\mathscr{R}_0$ dar vadinama lėtai kintančių *skirstinių klase. Visų reguliariai kintančių skirstinių klasė apibrėžiama $\mathscr{R} := \bigcup_{\alpha\geqslant 0} \mathscr{R}_\alpha$.*

Reguliariai kintančių funkcijų klasė buvo plačiai nagrinėta tikimybių teorijos ir matematinės analizės kontekstuose (žr. pvz.: [32, 96, 63, 104, 75, 89] ir [16] monografiją).

Glaudžiai susijusi klasė, apibendrinanti klasę $\mathscr{R}$, yra vadinamoji reguliariai kintančių plačiąja prasme skirstinių klasė.

Apibrėžimas 7.3. *P.f. F, turinti atramą $\mathbb{R}$, vadinama* turinčia reguliariai kintančią uodegą plačiąja prasme *(arba tiesiog* reguliariai kintančia plačiąja prasme*) su indeksais $0 \leqslant \alpha \leqslant \beta$, rašoma $F \in \mathscr{ERV}_{\{\alpha,\beta\}}$, jei bet kuriam $y > 1$ teisinga*

$$y^{-\beta} \leqslant \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant y^{-\alpha}.$$

Visų reguliariai kintančių plačiąja prasme skirstinių klasė apibrėžiama $\mathscr{ERV} := \bigcup_{0\leqslant\alpha\leqslant\beta} \mathscr{ERV}_{\{\alpha,\beta\}}$.

Galima nesunkiai pastebėti, kad jei $\alpha = \beta > 0$, tuomet $\mathscr{ERV}_{\{\alpha,\beta\}} = \mathscr{R}_\alpha$. Klasės $\mathscr{ERV}$ savybės buvo plačiau nagrinėtos [29, 67, 24, 121] darbuose (taip pat žr. ten nurodytą literatūrą).

Toliau apibrėžiama nuosaikiai kintančių skirstinių klasė, [29] straipsnyje pristatyta kaip klasės $\mathscr{R}$ apibendrinimas.

Apibrėžimas 7.4. *P.f. F, turinti atramą $\mathbb{R}$, vadinama* turinčia nuosaikiai kintančią uodegą *(arba* nuosaikiai kintančia*), rašoma $F \in \mathscr{C}$,* *jei*

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}=1.$$

Nuosaikiai kintančių skirstinių klasė sulaukė daug dėmesio ir buvo plačiai nagrinėta literatūroje (žr. pvz.: [31, 25, 23, 65, 5, 18, 117]).

Žemiau pateikiama dar platesnė dominuojančiai kintančių skirstinių klasė, kuri pirmą kartą buvo pristatyta [47] ir vėliau plačiai analizuota [57, 106, 113, 114, 56, 130, 132, 26] ir kt. darbuose.

Apibrėžimas 7.5. *P.f. F, turinti atramą $\mathbb{R}$, vadinama* turinčia dominuojančiai kintančią uodegą *(arba tiesiog* dominuojančiai kintančia*), rašoma, $F\in\mathscr{D}$, jei bet kuriam fiksuotam $y\in(0,1)$ teisinga*

$$\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}<\infty.$$

Toliau pristatomos dvi plačiai nagrinėjamos reguliarumo klasės, apibrėžtos Chistyakov [27] straipsnyje. Pirmoji jų – subeksponentinių skirstinių klasė. Svarbu atkreipti dėmesį, kad nors ši klasė iš pradžių buvo apibrėžta skirstiniams, kuriems teisinga $\lim_{x\uparrow 0}F(x)=0$, šioje disertacijoje naudojamas apibendrintas apibrėžimas, apimantis skirstinius turinčius atramą $\mathbb{R}$ (žr. [50, 3.2 skyrių]).

Apibrėžimas 7.6. *P.f. F, turinti atramą $\mathbb{R}$, vadinama* subeksponentine*, rašoma $F\in\mathscr{S}$, jei p.f. $F_+(x):=F(x)\mathbb{I}_{[0,\infty)}(x)$ yra teisinga*

$$\lim_{x\to\infty}\frac{\overline{F_+*F_+}(x)}{\overline{F}_+(x)}=2.$$

Subeksponentinių skirstinių klasė buvo plačiai nagrinėta literatūroje, pvz.: [28, 111, 57, 66, 107, 106, 56, 115, 129, 110]. Antroji, platesnė reguliarumo klasė, apimanti vadinamuosius ilgauodegius skirstinius, buvo tirta [42, 30, 66, 23, 125, 49] (taip pat žr. ten nurodytą literatūrą).

Apibrėžimas 7.7. *P.f. F, turinti atramą $\mathbb{R}$, vadinama* ilgauodege*, rašoma $F\in\mathscr{L}$, jei bet kuriam $y>0$ teisinga*

$$\lim_{x\to\infty}\frac{\overline{F}(x-y)}{\overline{F}(x)}=1.$$

Detalesnė subeksponentinių ir ilgauodegių skirstinių apžvalga pateikiama [50, 2 ir 3 skyriai].

Šiame skyrelyje apibrėžtų sunkiauodegių skirstinių poklasių tarpusavio ryšius galima apibendrinti žemiau pateikiamais įdėjimo sąryšiais (žr. pvz.: [57, 16, 23]):

$$\begin{gathered}\mathscr{R} \subset \mathscr{E}\mathscr{R}\mathscr{V} \subset \mathscr{C} \subset \mathscr{D} \cap \mathscr{L} \subset \mathscr{S} \subset \mathscr{L} \subset \mathscr{H}; \\ \mathscr{D} \subset \mathscr{H}; \qquad \mathscr{D} \setminus \mathscr{L} \neq \emptyset.\end{gathered} \tag{7.4}$$

Schematinė (7.4) įdėjimų iliustracija pateikiama 2.1 paveiksle.

Svarbu atkreipti dėmesį, kad visi įdėjimai, pateikiami (7.4) yra griežti. Cline ir Samorodnitsky [31] pateikė du pavyzdžius, kurie parodė $\mathscr{R} \subset \mathscr{E}\mathscr{R}\mathscr{V} \subset \mathscr{C}$ įdėjimų griežtumą. Peter ir Paul skirstinys, apibrėžtas 2.5 pavyzdyje priklauso klasei $\mathscr{D} \setminus \mathscr{L}$ (taip pat žr. [57]). Weibull skirstinys [57] straipsnyje buvo pateiktas kaip subeksponentinio, bet ne dominuojančiai kintančio skirstinio pavyzdys. Klasei $\mathscr{L} \setminus \mathscr{S}$ priklausančių skirstinių pavyzdžiai buvo sukonstruoti [42, 88], [50, 3.7 skyrius]. Daugiau sunkiauodegių skirstinių pavyzdžių taip pat galima rasti [45, 31] straipsniuose.

Sunkiauodegių skirstinių klasės ir pačių skirstinių uodegų asimptotinis elgesys gali būti charakterizuojamas specialiais indeksais. Vienas iš jų – vadinamasis *viršutinis Matuszewska indeksas*, pasiūlytas naudoti [81] ir vėliau sutinkamas įvairiuose sunkiauodegių skirstinių tyrimuose, pvz.: [16, 2 Skyrius], [31, 106, 25, 130, 132, 62] ir kt. P.f. F *viršutinis Matuszewska indeksas* J_F^+ apibrėžiamas

$$\begin{aligned}J_F^+ &:= \inf_{y>1} \left\{ -\frac{1}{\log y} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right\} \\ &= -\lim_{y\to\infty} \frac{1}{\log y} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.\end{aligned} \tag{7.5}$$

Kitas svarbus indeksas yra vadinamasis *L-indeksas*, naudotas pvz., [120, 131, 130, 132, 62, 76] darbuose. P.f. F *L-indeksas* L_F apibrėžiamas

$$L_F := \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}. \tag{7.6}$$

Iš reguliarumo klasių 7.2, 7.4, 7.5 apibrėžimų ir indeksų (7.5), (7.6) išplaukia (taip pat žr. [106])

$$\begin{aligned}
&F \in \mathscr{D} && \iff && J_F^+ < \infty && \iff && L_F > 0;\\
&F \in \mathscr{C} && \iff && L_F = 1 && \implies && J_F^+ < \infty;\\
&F \in \mathscr{R}_\alpha && \implies && L_F = 1, \quad J_F^+ = \alpha.
\end{aligned}$$

Šioje disertacijoje taip pat nagrinėjama viena papildoma funkcijų klasė, kurios elementai tenkina tam tikrą asimptotinio monotoniškumo savybę.

Apibrėžimas 7.8. *P.f. F, turinti atramą $\mathbb{R}$, vadinama* turinčia teigiamai gęstančią uodegą*, rašoma $F \in \mathscr{P}_{\mathscr{D}}$, jei bet kuriam fiksuotam $y \in (0,1)$ yra teisinga*

$$\liminf_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} > 1.$$

Teigiamai gęstančias uodegas turinčių skirstinių klasė buvo nagrinėta [16, 75, 69] darbuose. Galima nesunkiai įsitikinti, jog

$$\mathscr{R}_+ := \bigcup_{\alpha>0} \mathscr{R}_\alpha \subset \mathscr{P}_{\mathscr{D}}; \quad \mathscr{R}_0 \cap \mathscr{P}_{\mathscr{D}} = \emptyset; \quad \mathscr{C} \cap \mathscr{P}_{\mathscr{D}} \neq \emptyset; \quad \mathscr{P}_{\mathscr{D}} \setminus \mathscr{H} \neq \emptyset.$$

7.2.2 Priklausomybės struktūros

Kadangi disertacijoje nagrinėjami ne tik nepriklausomi a.d.s, šiame skyrelyje pristatoma pagrindinė darbe naudojama priklausomybės struktūra – vadinamasis *kvazi-asimptotinis nepriklausomumas*, pirmą kartą apibrėžtas Chen ir Yuen [25].

Apibrėžimas 7.9 (pQAI struktūra)**.** *Realieji a.d.s $\xi_1,\ldots,\xi_n$, $n \in \mathbb{N}$, su p.f.s, turinčiomis atramą $\mathbb{R}$, yra vadinami poromis* kvazi-asimptotiškai nepriklausomais (pQAI)*, jei visoms indeksų poroms $k, l \in \{1,\ldots,n\}$, $k \neq l$, teisinga*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^+ > x)}{\overline{F}_{\xi_k^+}(x) + \overline{F}_{\xi_l^+}(x)} = \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^- > x)}{\overline{F}_{\xi_k^+(x)} + \overline{F}_{\xi_l^+}(x)} = 0. \tag{7.7}$$

Svarbu atkreipti dėmesį, kad ši priklausomybės struktūra apibrėžiama per asimptotinį skirstinių u.f.s elgesį – laikoma, kad a.d.s ekstremalių reikšmių srityse elgiasi panašiai kaip nepriklausomi a.d.s. Panašios, bet griežtesnės (žr. 2.1 teiginį) priklausomybės struktūros taip pat buvo nagrinėtos kitų autorių darbuose – pavyzdžiui: striprus kvazi-asimptotinis nepriklausomumas (pSQAI) [56, 77], kvazi-asimptotinis nepriklausomumas uodegoje (pTQAI) [26], asimptotinis nepriklausomumas (pAI)[1] [26], prielaida B [76].

Priklausomybės struktūros tarp a.d.s gali būti modeliuojamos pasinaudojant kopulomis ir Sklar'o teorema.

Apibrėžimas 7.10. *(Žr. [84, 2.2.2 apibrėžimas]) Dvimate kopula vadinama funkcija $C\colon [0{,}1]^2 \to [0{,}1]$, kuri tenkina savybes:*

- $C(x,0) = C(0,x) = 0 \qquad$ *visiems* $\qquad x \in [0{,}1]$;
- $C(x,1) = C(1,x) = x \qquad$ *visiems* $\qquad x \in [0{,}1]$;
- *visiems* $0 \leqslant u_1 \leqslant u_2 \leqslant 1$ *ir* $0 \leqslant v_1 \leqslant v_2 \leqslant 1$,

$$C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geqslant 0.$$

Teorema 7.1 (Sklar'o teorema, žr. [84, 2.3.3 ir 2.4.1 teoremos])**.** *Tegu F yra dvimatė p.f. su marginaliosiomis pasiskirstymo funkcijomis (p.f.s) F_1 ir F_2. Tuomet egzistuoja dvimatė kopula C, tokia, kuriai*

$$F(x_1,x_2) = C\left(F_1(x_1), F_2(x_2)\right) \quad \textit{visiems} \quad x_1,x_2 \in \overline{\mathbb{R}}. \tag{7.8}$$

Jei p.f.s F_1 ir F_2 yra tolydžios, tuomet kopula C yra vienintelė. Priešingu atveju, kopula C yra vienintelė srityje $\mathrm{Ran}(F_1) \times \mathrm{Ran}(F_2)$*. Iš kitos pusės, tegu C yra dvimatė kopula, o F_1, F_2 – dvi p.f.s. Tuomet egzistuoja dvimatė p.f. F, apibrėžta* (7.8)*, su marginaliosiomis p.f.s F_1 ir F_2.*

Galima parodyti, kad pQAI priklausomybės struktūros atveju, a.d.s ξ_1, ξ_2 su p.f.s, turinčiomis atramą $\mathbb{R}$ yra pQAI, jei atsitiktinio vektoriaus (ξ_1, ξ_2) dvimatė p.f. apibrėžiama (7.8) su viena iš žemiau pateikiamų dvimačių kopulų:

- *Frank* kopula [84, 4.2 skyrius] su parametru $\vartheta \in \mathbb{R} \setminus \{0\}$:

$$C_\vartheta(u,v) = -\frac{1}{\vartheta}\log\left(1 + \frac{(\mathrm{e}^{-\vartheta u}-1)(\mathrm{e}^{-\vartheta v}-1)}{\mathrm{e}^{-\vartheta}-1}\right), \quad u,v \in [0,1].$$

[1] Asimptotinio nepriklausomumo (pAI) prielaida yra griežtesnė už kvazi-asimptotinio nepriklausomumo prielaidą, jei a.d.s $\xi_1,\ldots,\xi_n$ teisinga $\lim_{x\to\infty} \overline{F}_{\xi_k^-}(x)/\overline{F}_{\xi_k}(x) = 0$, $k = 1,\ldots,n$. Ši papildoma prielaida taip pat buvo naudota [26].

- *Ali-Mikhail-Haq* (AMH) kopula [4] su parametru $\vartheta \in (-1, 1)$:

$$C_\vartheta(u,v) = \frac{uv}{1 - \vartheta(1-u)(1-v)}, \quad u, v \in [0,1].$$

- *Farlie-Gumbel-Morgenstern* (FGM) kopula [84, 3.12 pavyzdys] su parametru $\vartheta \in [-1, 1]$:

$$C_\vartheta(u,v) = uv + \vartheta uv(1-u)(1-v), \quad u, v \in [0,1].$$

7.2.3 Kvantilių funkcijos

Šiame skyrelyje apibrėžiamos pasiskirstymo funkcijų apibendrintos atvirkštinės funkcijos (arba, kitaip, kvantilių funkcijos), kurios disertacijoje naudojamos tiek Monte Carlo simuliacijų metu generuojant a.d.s realizacijas, tiek apibrėžiant ir analizuojant rizikos matus.

Apibrėžimas 7.11. *Realiojo a.d. X p.f. F* apibendrinta atvirkštine *arba* kvantilių funkcija *yra vadinama funkcija $F^{\leftarrow}\colon [0,1] \to \overline{\mathbb{R}}$, apibrėžta*

$$F^{\leftarrow}(q) := \inf\left\{x \in \mathbb{R} : F(x) \geqslant q\right\} = \inf\left\{x \in \mathbb{R} : \overline{F}(x) \leqslant 1-q\right\}.$$

Laikoma, kad $F^{\leftarrow}(q) = \infty$, jei $\{x \in \mathbb{R} : F(x) \geqslant q\} = \emptyset$.

Toliau pateikiama pastaba apibrėžia u.f.s atvirkštines funkcijas. Jų apibrėžimas tiesiogiai išplaukia iš 7.11 apibrėžimo.

Pastaba 7.1. *U.f. $\overline{F}$ apibendrinta atvirkštinė funkcija yra $\overline{F}^{\leftarrow} : [0,1] \to \overline{\mathbb{R}}$, apibrėžta*

$$\overline{F}^{\leftarrow}(p) = \inf\left\{x \in \mathbb{R} : \overline{F}(x) \leqslant p\right\}.$$

Laikoma, kad $\overline{F}^{\leftarrow}(p) = \infty$, jei $\{x \in \mathbb{R} : \overline{F}(x) \leqslant p\} = \emptyset$.

7.2.4 Rizikos matai

Šiame skyrelyje pateikiami pagrindiniai rizikos matų apibrėžimai ir savybės, taip pat pristatomi kai kurių rizikos matų pavyzdžiai.

Griežtai matematiniu požiūriu rizikos matai yra realias reikšmes įgyjantys funkcionalai, apibrėžti pasirinktoje a.d.s klasėje.

Apibrėžimas 7.12. Rizikos matu *vadinamas atvaizdis $\rho\colon \mathcal{G} \to \mathbb{R}$. Čia $\mathcal{G}$ yra nagrinėjamų finansinių pozicijų (atsitiktinių dydžių) klasė.*

Praktikoje, rizikos mato ρ ir klasės $\mathcal{G}$ interpretacija priklauso nuo taikymo srities, kurioje jie yra naudojami. Pavyzdžiui, finansų taikymuose pozicija $X \in \mathcal{G}$ gali būti diskontuota portfelio vertė ar jo generuotas pelnas. Tokiu atveju, rizika $\rho(X)$ galėtų atspindėti minimalų kapitalo kiekį, kurį reikėtų pridėti, kad pozicija taptų priimtina [9]. Draudimo kontekste, a.d.s iš klasės $\mathcal{G}$ dažnai atspindi potencialius nuostolius, kuriuos gali patirti draudikas žalos padengimo atveju, o $\rho(X)$ – draudimo įmokos dydį [58].

Svarbu atkreipti dėmesį, kad, nemažinant bendrumo, finansinės pozicijos iš klasės $\mathcal{G}$ gali būti analizuojamos tiek pelno-nuostolio, tiek nuostolio-pelno skalėse. Toliau disertacijoje finansinės pozicijos bus nagrinėjamos nuostolio-pelno skalėje.

Artzner su bendraautoriais [9] darbe įvedė vadinamąją *suderinto* rizikos mato sąvoką, kuri apibrėžė finansine logika pagrįstas aksiomas tenkinančius rizikos matus.

Apibrėžimas 7.13. *Rizikos matas $\rho\colon \mathcal{G} \to \mathbb{R}$ vadinamas* suderintu*, jei jis tenkina žemiau pateikiamas aksiomas:*

- **Invariantiškumas poslinkiui.** *Visiems $X \in \mathcal{G}$ ir $\alpha \in \mathbb{R}$,*
$$\rho(X + \alpha) = \rho(X) + \alpha.$$
- **Teigiamas homogeniškumas.** *Visiems $X \in \mathcal{G}$ ir $\lambda \geqslant 0$,*
$$\rho(\lambda X) = \lambda\rho(X).$$
- **Subadityvumas.** *Visiems $X_1, X_2 \in \mathcal{G}$,*
$$\rho(X_1 + X_2) \leqslant \rho(X_1) + \rho(X_2).$$
- **Monotoniškumas.** *Visiems $X, Y \in \mathcal{G}$, tokiems, kad $X \leqslant Y$,*
$$\rho(X) \leqslant \rho(Y).$$

Vienas žinomiausių rizikos matų – *vertės pokyčio rizika* (VaR). Šis rizikos matas su pasikliovimo lygmeniu $q \in (0, 1)$ finansinei pozicijai

$X \in \mathcal{G}$ apibrėžiamas

$$\mathrm{VaR}_q(X) := \inf\{x \in \mathbb{R} : \mathbb{P}(X > x) \leqslant 1 - q\} = F_X^{\leftarrow}(q).$$

Galima įsitikinti [9], kad VaR nėra suderintas rizikos matas, nes netenkina subadityvumo aksiomos. Be to, jis yra invariantiškas nuostoliams virš pasirinkto pasikliovimo lygmens [126]. Kaip suderintą VaR alternatyvą, Artzner ir bendraautoriai [9] pasiūlė naudoti *deficito vidurkį* (ES). ES su pasikliovimo lygmeniu $q \in (0, 1)$ pozicijai $X \in \mathcal{G}$, $\mathbb{E}X^+ < \infty$, apibrėžiamas

$$\mathrm{ES}_q(X) := \frac{1}{1-q} \int_q^1 \mathrm{VaR}_u(X) \mathrm{d}u.$$

Galima įsitikinti, kad deficito vidurkis yra ekvivalentus *sąlyginei vertės pokyčio rizikai* (CVaR) – rizikos matui, pasiūlytam [87, 94, 93]. CVaR su lygmeniu $q \in (0, 1)$ pozicijai $X \in \mathcal{G}$, $\mathbb{E}X^+ < \infty$, apibrėžiamas

$$\mathrm{CVaR}_q(X) := \inf_{x \in \mathbb{R}} \left\{ x + \frac{\mathbb{E}(X - x)^+}{1 - q} \right\}.$$

Tolesnėje skyrelio dalyje pristatomas šioje disertacijoje plačiai nagrinėjamas Haezendonck-Goovaerts (HG) rizikos matas, kuris buvo pavadintas jį 1982 metais pasiūliusių autorių garbei [60]. Formaliai, HG rizikos matai sudaro rizikos matų šeimą, kurios nariai apibrėžiami jų elgesį nusakančiomis normalizuotomis Young funkcijomis.

Apibrėžimas 7.14. *Funkcija* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *yra vadinama* normalizuota Young funkcija*, jei* φ *yra neneigiama ir iškila intervale* $[0, \infty)$ *ir tenkina* $\varphi(0) = 0$*,* $\varphi(1) = 1$*,* $\varphi(\infty) = \infty$*.*

Prieš pateikiant HG rizikos mato apibrėžimą, svarbu aptarti finansinių pozicijų aibę $\mathcal{G}$, kurioje šis matas yra apibrėžtas. Štai [15, 108, 109] straipsniuose HG rizikos matas buvo nagrinėjamas vadinamosiose Orlicz erdvėse, kurios sulaukė didelio dėmesio rizikos matų kontekste [15].

Apibrėžimas 7.15. *Realiųjų a.d.s* Orlicz erdvė $\mathbb{L}^\varphi$ *ir* Orlicz širdis $\mathbb{L}_0^\varphi$*, susietos su funkcija* $\varphi\colon \mathbb{R} \to \mathbb{R}$*, atitinkamai apibrėžiamos, kaip*

$$\mathbb{L}^\varphi = \Big\{ X : \mathbb{E}\big(\varphi(c|X|)\big) < \infty \text{ kažkokiam } c > 0 \Big\},$$
$$\mathbb{L}_0^\varphi = \Big\{ X : \mathbb{E}\big(\varphi(c|X|)\big) < \infty \text{ visiems } c > 0 \Big\}.$$

Priklausomai nuo funkcijos φ pasirinkimo, erdvės $\mathbb{L}^{\varphi}$ ir $\mathbb{L}_0^{\varphi}$ gali sutapti. Funkcijos, kuriai teisinga $\mathbb{L}^{\varphi} = \mathbb{L}_0^{\varphi}$, pavyzdys – laipsninė funkcija $\varphi(t) = t^{\varkappa}$, $\varkappa \geqslant 1$.

Toliau pateikiamas HG rizikos mato apibrėžimas.

Apibrėžimas 7.16 (Haezendonck-Goovaerts rizikos matas)**.** *Tegu φ yra normalizuota Young funkcija, $q \in (0,1)$ ir $X \in \mathbb{L}^{\varphi}$.* Haezendonck-Goovaerts (HG) rizikos matas *atsitiktiniam dydžiui X apibrėžiamas*

$$\mathrm{HG}_q(X) = \inf_{x \in \mathbb{R}} \{x + h(x,q)\}.$$

Čia $h = h(x,q)$ yra lygties

$$\mathbb{E}\varphi\left(\frac{(X-x)^+}{h}\right) = 1 - q \tag{7.9}$$

sprendinys, jei $\overline{F}_X(x) > 0$, ir $h(x,q) = 0$, jei $\overline{F}_X(x) = 0$.

Galima parodyti [15, 3 teiginys (b, d)], kad HG rizikos mato lygtį (2.41) minimizuojantis taškas (x_*, h_*) egzistuoja visiems $q \in (0,1)$ ir yra vienintelis, jei φ yra griežtai iškila, o $X \in \mathbb{L}_0^{\varphi}$. Be to, HG rizikos matas yra suderintas rizikos matas [15, 3 teiginys (e)].

Atskiru atveju, kai $\varphi(t) = t$, HG rizikos matas yra ekvivalentus CVaR ir ES rizikos matams.

7.3 Normaliųjų atsitiktinių dydžių sandaugų skirstinio uodegos asimptotika

Kartu su tiksliomis normaliųjų a.d.s sandaugos skirstinio formulėmis neretai nagrinėjama u.f.s asimptotika. Arendarczyk ir Dębicki [6] straipsnyje gavo asimptotinį rezultatą skirstiniams su Weibull tipo uodegomis. Pasirodo, kad šis rezultatas taip pat pritaikomas ir normaliųjų a.d.s sandaugoms.

Teorema 7.2 (Žr. [6, 2.1 lema])**.** *Tegu ξ_1 ir ξ_2 yra nepriklausomi ir neneigiami a.d.s, tokie, kad*

$$\overline{F}_{\xi_1}(x) \underset{x\to\infty}{\sim} A_1 x^{\gamma_1} \exp\{-\beta_1 x^{\alpha_1}\}, \quad \overline{F}_{\xi_2}(x) \underset{x\to\infty}{\sim} A_2 x^{\gamma_2} \exp\{-\beta_2 x^{\alpha_2}\}.$$

Čia $A_i > 0$, $\gamma_i \in \mathbb{R}$, $\beta_i > 0$, $\alpha_i > 0$, $i = 1, 2$. *Tuomet*

$$\overline{F}_{\xi_1\xi_2}(x) \underset{x\to\infty}{\sim} Ax^{\gamma} \exp\{-\beta x^{\alpha}\}$$

ir

$$\begin{aligned}
\alpha &= \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \\
\beta &= \beta_1^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \beta_2^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \left(\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} + \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \right), \\
\gamma &= \frac{\alpha_1\alpha_2 + 2\alpha_1\gamma_2 + 2\alpha_2\gamma_1}{2(\alpha_1+\alpha_2)}, \\
A &= \sqrt{2\pi}\, \frac{A_1A_2}{\sqrt{\alpha_1+\alpha_2}}\, (\alpha_1\beta_1)^{\frac{\alpha_2-2\gamma_1+2\gamma_2}{2(\alpha_1+\alpha_2)}}\, (\alpha_2\beta_2)^{\frac{\alpha_1-2\gamma_2+2\gamma_1}{2(\alpha_1+\alpha_2)}}\,.
\end{aligned}$$

Pasinaudojus matematinės indukcijos metodu ir 7.2 teorema galima parodyti, kad galioja toks teiginys.

Teiginys 7.1. *Tegu* $\xi_1, \xi_2, \ldots, \xi_n$, $n \in \mathbb{N}$, *yra nepriklausomi a.d.s, tokie, kad visiems* k, $\xi_k \overset{d}{=} \Phi$, *ir* $\Pi_n := \prod_{k=1}^n \xi_k$. *Tuomet*

$$\overline{F}_{\Pi_n}(x) \underset{x\to\infty}{\sim} \frac{2^{(n/2)-1}}{\sqrt{\pi n}} x^{-\frac{1}{n}} \exp\left\{-\frac{n}{2}x^{\frac{2}{n}}\right\}.$$

Asimptotinę formulę pateiktą 7.1 teiginyje galima nesunkiai apibendrinti nebūtinai vienodai pasiskirsčiusiems normaliesiems a.d.s su nuliniu vidurkiu.

Išvada 7.1. *Tegu* $\xi_1, \xi_2, \ldots, \xi_n$, $n \in \mathbb{N}$, *yra nepriklausomi a.d.s, tokie, kad visiems* k, $\xi_k \overset{d}{=} \Phi_{\{0,\sigma_k\}}$, *ir* $\widehat{\Pi}_n := \prod_{k=1}^n \xi_k$. *Tuomet*

$$\overline{F}_{\widehat{\Pi}_n}(x) \underset{x\to\infty}{\sim} \frac{2^{(n/2)-1}(\sigma^{(n)})^{1/n}}{\sqrt{\pi n}} x^{-\frac{1}{n}} \exp\left\{-\frac{n(\sigma^{(n)})^{-2/n}}{2}\, x^{\frac{2}{n}}\right\}.$$

Čia $\sigma^{(n)} := \prod_{k=1}^n \sigma_k$.

Pastaba 7.2. *Asimptotinė formulė 7.1 išvadoje išplaukia tiesiai iš 7.1 teiginio, nes visiems* k, *a.d.* $\xi_k = \widehat{\xi}_k/\sigma_k \overset{d}{=} \Phi$, *ir*

$$\overline{F}_{\widehat{\Pi}_n}(x) = \mathbb{P}\left(\widehat{\Pi}_n > x\right) = \mathbb{P}\left(\Pi_n > \frac{x}{\sigma^{(n)}}\right) = \overline{F}_{\Pi_n}\left(\frac{x}{\sigma^{(n)}}\right).$$

Svarbu atkreipti dėmesį, kad, priklausomai nuo daugiklių skaičiaus, sandauga gali būti lengvauodegė arba sunkiauodegė.

Lema 7.1. *Tarkime, kad* $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, *yra nepriklausomi a.d.s, tokie, kad* $\xi_k \stackrel{d}{=} \Phi_{\{0,\sigma_k\}}$, $k = 1, \ldots, n$, *ir* $\widehat{\Pi}_n := \prod_{k=1}^n \xi_k$. *Sandauga* $\widehat{\Pi}_n$ *yra lengvauodegė, jei* $n \in \{1, 2\}$, *ir sunkiauodegė, jei* $n > 2$.

Tolesnėje skyrelio dalyje pateikiami pagrindiniai disertacijos rezultatai, kurie patikslina 7.1 teiginio asimptotinius įverčius. Pirmiausia, 7.3 teoremoje gaunama asimptotinė formulė su pirmos eilės liekamuoju nariu standartinių normaliųjų a.d.s sandaugoms, o 7.2 išvada ją apibendrina nulinio vidurkio normaliųjų a.d.s sandaugoms.

Teorema 7.3 (Žr. [73, 2.2 teorema]). *Tegu* $\xi_1, \xi_2, \ldots, \xi_n$, $n \in \mathbb{N}$, *yra nepriklausomi a.d.s, tokie, kad visiems* k, $\xi_k \stackrel{d}{=} \Phi$, *ir* $\Pi_n := \prod_{k=1}^n \xi_k$. *Tuomet*

$$\overline{F}_{\Pi_n}(x) = \frac{2^{(n/2)-1}}{\sqrt{\pi n}} \, x^{-\frac{1}{n}} \exp\left\{-\frac{n}{2}\, x^{\frac{2}{n}}\right\} \left(1 + O_n\left(x^{-\frac{2}{n}}\right)\right),$$

kai $x \to \infty$. *Čia konstanta iš* O_n *priklauso nuo* n.

Pritaikius tą patį samprotavimą kaip ir 7.2 pastaboje, gaunama toliau pateikiama išvada.

Išvada 7.2. *Tegu* $\xi_1, \xi_2, \ldots, \xi_n$, $n \in \mathbb{N}$, *yra nepriklausomi a.d.s, tokie, kad visiems* k, $\xi_k \stackrel{d}{=} \Phi_{\{0,\sigma_k\}}$, *ir* $\widehat{\Pi}_n := \prod_{k=1}^n \xi_k$. *Tuomet*

$$\begin{aligned}\overline{F}_{\widehat{\Pi}_n}(x) = {} & \frac{2^{(n/2)-1}(\sigma^{(n)})^{1/n}}{\sqrt{\pi n}} x^{-\frac{1}{n}} \\ & \times \exp\left\{-\frac{n(\sigma^{(n)})^{-2/n}}{2}\, x^{\frac{2}{n}}\right\} \left(1 + O_n\left(x^{-\frac{2}{n}}\right)\right)\end{aligned}$$

kai $x \to \infty$. *Čia* $\sigma^{(n)} := \prod_{k=1}^n \sigma_k$, *o konstanta iš* O_n *priklauso nuo* n *ir* $\sigma^{(n)}$.

7.4 Atsitiktinių sumų su sunkiauodegiais dėmenimis apibendrintų momentų asimptotinės savybės

Vienas pagrindinių rezultatų, kuris paskatino tolesnius šioje disertacijoje aprašomus a.d.s sumų apibendrintų uodegos momentų asimptotikos tyrimus, buvo gautas Leipaus ir bendraautorių [76] straipsnyje. Minėtame darbe buvo gauti asimptotiniai rėžiai sumoms $S_n^{\theta\xi}$ transformuotoms laipsnine funkcija $\varphi(x) = x^m$, $m \in \mathbb{N}$, atveju, kai pagrindiniai a.d.s turi dominuojančiai kintančias uodegas ir tenkina priklausomybės struktūrą, [76] pavadintą prielaida B.

Apibrėžimas 7.17 (Prielaida B)**.** *Realieji a.d.s $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, su p.f.s turinčiomis atramą $\mathbb{R}$, tenkina* prielaidą B, *jei visoms indeksų poroms $k, l \in \{1, \ldots, n\}$, $k \neq l$, teisinga*

$$\lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}\left(\xi_k^+ > x \mid \xi_l^+ > u\right) = \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}\left(\xi_k^- > x \mid \xi_l^+ > u\right)$$
$$= \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}\left(\xi_k^+ > x \mid \xi_l^- > u\right) = 0.$$

Teorema 7.4 (Žr. [76, 5 teorema])**.** *Tegu $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, yra realieji a.d.s, tenkinantys prielaidą B, tokie, kad $F_{\xi_1} \in \mathscr{D}$, $\mathbb{E}\,|\xi_1|^m < \infty$ kažkokiam $m \in \mathbb{N}$, o $\theta_1, \ldots, \theta_n$ yra neneigiami, neišsigimę nulyje, aprėžti a.d.s, nepriklausomi nuo $\xi_1, \ldots, \xi_n$. Jei*

$$\overline{F}_{\theta_k\xi_k}(x) \asymp \overline{F}_{\theta_1\xi_1}(x), \quad \overline{F}_{\theta_k\xi_k^-}(x) = O\left(\overline{F}_{\theta_1\xi_1}(x)\right),$$

visiems $k = 2, \ldots, n$, tuomet

$$L_n^\xi \sum_{k=1}^n \mathbb{E}\left((\theta_k\xi_k)^m \mathbb{I}_{\{\theta_k\xi_k > x\}}\right) \underset{x\to\infty}{\lesssim} \mathbb{E}\left((S_n^{\theta\xi})^m \mathbb{I}_{\{S_n^{\theta\xi} > x\}}\right)$$
$$\underset{x\to\infty}{\lesssim} \frac{1}{L_n^\xi} \sum_{k=1}^n \mathbb{E}\left((\theta_k\xi_k)^m \mathbb{I}_{\{\theta_k\xi_k > x\}}\right).$$

Kiek vėliau Sprindys ir Šiaulys [105] straipsnyje gavo tikslius asimptotinius sąryšius sumoms S_n^ξ su nuosaikiai kintančiais dėmenimis. Šį kartą vietoje prielaidos B buvo naudojama platesnė pQAI priklausomybės struktūra, o laipsninės funkcijos laipsnis galėjo įgyti ne tik natūralias reikšmes. Autoriai taip pat parodė, kad sumų S_n^ξ apibendrintų uodegos momentų asimptotinis elgesys priklauso nuo sunkiauodegių sumos dėmenų iš klasės $\mathscr{C}$, bet ne nuo asimptotiškai lengvesnes uodegas turinčių a.d.s.

Teorema 7.5 (Žr. [105, 4 teorema])**.** *Tegu $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, yra realieji* pQAI *a.d.s, tokie, kad $F_{\xi_L} \in \mathscr{C}$ indeksui $L \in \{1, \ldots, n\}$, o atvejais, kai $k \neq L$, teisinga $F_{\xi_k} \in \mathscr{C}$ arba $\mathbb{P}(|\xi_k| > x) = o(\overline{F}_{\xi_L}(x))$. Jei*

$\max\limits_{1\leqslant k\leqslant n} \mathbb{E}(\xi_k^+)^\alpha < \infty$ *kažkokiam* $\alpha \geqslant 0$*, tuomet visiems* $\beta \in [0,\alpha]$ *teisinga*

$$\mathbb{E}\Big((S_n^\xi)^\beta \mathbb{I}_{\{S_n^\xi > x\}}\Big) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k^\beta\, \mathbb{I}_{\{\xi_k > x\}}\Big),$$

o visiems $\beta \in (0,\alpha]$ *teisinga*

$$\mathbb{E}\left(\Big((S_n^\xi - x)^+\Big)^\beta\right) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}_n} \mathbb{E}\Big((\xi_k - x)^+\Big)^\beta.$$

Čia indeksų aibė $\mathcal{I}_n \subseteq \{1,\ldots,n\}$ *yra tokia, kad* $F_{\xi_k} \in \mathscr{C}$ *visiems* $k \in \mathcal{I}_n$.

Toliau pateikiami pagrindiniai šios disertacijos rezultatai, kurie apibrėžia atsitiktinių sumų su dominuojančiai kintančiais dėmenimis apibendrintų momentų asimptotinį elgesį. Pirmoji teorema įtvirtina asimptotinius rėžius sumoms S_n^ξ, o antroji juos apibendrina atsitiktinėms svorinėms sumoms $S_n^{\theta\xi}$.

Teorema 7.6 (Žr. [39, 2 teorema]). *Tegu* $\xi_1,\ldots,\xi_n$*,* $n\in\mathbb{N}$*, yra realieji* pQAI *a.d.s, tokie, kad* $F_{\xi_k} \in \mathscr{D}$ *visiems* $k \in \{1,\ldots,n\}$*, o* $\varphi : \mathbb{R} \to \mathbb{R}_+$ *yra funkcija, tenkinanti žemiau pateikiamas sąlygas:*

(i) φ *yra galiausiai*[2] *nemažėjanti;*

(ii) φ *yra galiausiai subhomogeniška, t. y.,* $\varphi(2x) \leqslant c\varphi(x)$ *kažkokiam* $c > 0$ *ir visiems pakankamai dideliems* x*;*

(iii) φ *yra galiausiai diferencijuojama;*

(iv) *visiems* $k \in \{1,2,\ldots,n\}$ *apibendrintas momentas* $\mathbb{E}(\varphi(\xi_k)\mathbb{I}_{\{\xi_k>x\}})$ *yra galiausiai baigtinis.*

Tuomet

$$\begin{aligned}\sum_{\substack{k=1\\ *}}^{n} L_{F_{\xi_k}} \mathbb{E}\left(\varphi(\xi_k)\mathbb{I}_{\{\xi_k>x\}}\right) &\underset{x\to\infty}{\lesssim} \mathbb{E}\left(\varphi(S_n^\xi)\mathbb{I}_{\{S_n^\xi>x\}}\right)\\ &\underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\left(\varphi(\xi_k)\mathbb{I}_{\{\xi_k>x\}}\right).\end{aligned}$$

Teorema 7.7 (Žr. [39, 3 teorema]). *Tegu* $\xi_1,\ldots,\xi_n$*,* $n\in\mathbb{N}$*, yra realieji* pQAI *a.d.s, tokie, kad* $F_{\xi_k} \in \mathscr{D}$ *visiems* $k \in \{1,\ldots,n\}$*, o* $\theta_1,\ldots,\theta_n$

[2] Laikoma, kad savybė A galioja *galiausiai*, jei ji galioja visiems pakankamai dideliems x.

yra galimai priklausomi, neneigiami ir neišsigimę nulyje a.d.s, kuriems teisinga

$$\max\{\mathbb{E}\theta_1^p,\ldots,\mathbb{E}\theta_n^p\}<\infty \quad \textit{kažkokiam} \quad p>\max\{J^+_{F_{\xi_1}},\ldots,J^+_{F_{\xi_n}}\}. \tag{7.10}$$

Taip pat tegu rinkiniai $\{\xi_1,\ldots,\xi_n\}$ *ir* $\{\theta_1,\ldots,\theta_n\}$ *yra nepriklausomi, o* $\varphi:\mathbb{R}\to\mathbb{R}_+$ *yra funkcija, tenkinanti sąlygas* (i), (ii), (iii) *iš 7.6 teoremos ir žemiau pateikiamą sąlygą:*

(iv)* *apibendrintas momentas* $\mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k>x\}}\right)$ *yra galiausiai baigtinis visiems* $k\in\{1,\ldots,n\}$.
Tuomet

$$\begin{aligned}\sum_{k=1}^n L_{F_{\xi_k}}\mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k>x\}}\right) &\underset{x\to\infty}{\lesssim} \mathbb{E}\left(\varphi(S_n^{\theta\xi})\mathbb{I}_{\{S_n^{\theta\xi}>x\}}\right)\\ &\underset{x\to\infty}{\lesssim} \sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}}\mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k>x\}}\right).\end{aligned} \tag{7.11}$$

Atskiras 7.6 ir 7.7 teoremų atvejis, kai φ – laipsninė funkcija.

Išvada 7.3. *Tegu* $\xi_1,\ldots,\xi_n$, $n\in\mathbb{N}$, *yra realieji* pQAI *a.d.s, tokie, kad* $F_{\xi_k}\in\mathscr{D}$ *visiems* $k\in\{1{,}2,\ldots,n\}$, *o* $\theta_1,\ldots,\theta_n$ *yra galimai priklausomi, neneigiami ir neišsigimę nulyje a.d.s, kuriems teisinga* (7.10). *Jei a.d.s rinkiniai* $\{\xi_1,\ldots,\xi_n\}$ *ir* $\{\theta_1,\ldots,\theta_n\}$ *yra nepriklausomi, o* $\mathbb{E}(\theta_k|\xi_k|)^\alpha<\infty$ *visiems* $k\in\{1{,}2,\ldots,n\}$ *ir kažkokiam* $\alpha\in[0,\infty)$, *tuomet*

$$\begin{aligned}\sum_{k=1}^n L_{F_{\xi_k}}\mathbb{E}\left((\theta_k\xi_k)^\alpha\mathbb{I}_{\{\theta_k\xi_k>x\}}\right) &\underset{x\to\infty}{\lesssim} \mathbb{E}\left((S_n^{\theta\xi})^\alpha\mathbb{I}_{\{S_n^{\theta\xi}>x\}}\right)\\ &\underset{x\to\infty}{\lesssim} \sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}}\mathbb{E}\left((\theta_k\xi_k)^\alpha\mathbb{I}_{\{\theta_k\xi_k>x\}}\right).\end{aligned}$$

Skyrelio pabaigoje pateikiami keli pastebėjimai.

Pastaba 7.3. *Vietoje klasės* $\mathscr{D}$ *nagrinėjant siauresnę skirstinių klasę* $\mathscr{C}$, (7.11) *gaunamas asimptotinio ekvivalentumo sąryšis. Tiksliau, jei* $F_{\xi_k}\in\mathscr{C}$ *visiems* $k\in\{1,\ldots,n\}$ *ir galioja visos kitos 7.7 teoremos sąlygos, tuomet*

$$\sum_{k=1}^n \mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k>x\}}\right) \underset{x\to\infty}{\sim} \mathbb{E}\left(\varphi(S_n^{\theta\xi})\mathbb{I}_{\{S_n^{\theta\xi}>x\}}\right).$$

Pateiktas sąryšis tiesiogiai išplaukia iš L-indekso ir klasės $\mathscr{C}$ *apibrėžimų.*

Pastaba 7.4. *Atvejais, kai $\alpha = 0$ 7.3 išvadoje arba $\varphi \equiv 1$ 7.7 teoremoje, gaunami žemiau pateikiami asimptotiniai rėžiai:*

$$\begin{aligned} &\sum_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\theta_k\xi_k}(x) \leqslant \sum_{k=1}^{n} L_{F_{\theta_k\xi_k}} \overline{F}_{\theta_k\xi_k}(x) \underset{x\to\infty}{\lesssim} \mathbb{P}\left(S_n^{\theta\xi} > x\right), \\ &\sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \overline{F}_{\theta_k\xi_k}(x) \geqslant \sum_{k=1}^{n} \frac{1}{L_{F_{\theta_k\xi_k}}} \overline{F}_{\theta_k\xi_k}(x) \underset{x\to\infty}{\gtrsim} \mathbb{P}\left(S_n^{\theta\xi} > x\right). \end{aligned} \tag{7.12}$$

Pastaba 7.5. *Tomis pačiomis sąlygomis kaip 7.7 teoremoje galima gauti asimptotinius rėžius sąlyginiam vidurkiui* $\mathbb{E}(\varphi(S_n^{\theta\xi}) \mid S_n^{\theta\xi} > x)$. *Pasinaudojus* (7.11) *ir* (7.12) *gaunama*

$$\begin{aligned} &\frac{\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k > x\}})}{\sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \overline{F}_{\theta_k\xi_k}(x)} \underset{x\to\infty}{\lesssim} \mathbb{E}\left(\varphi(S_n^{\theta\xi}) \mid S_n^{\theta\xi} > x\right), \\ &\frac{\sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}(\varphi(\theta_k\xi_k)\mathbb{I}_{\{\theta_k\xi_k > x\}})}{\sum_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\theta_k\xi_k}(x)} \underset{x\to\infty}{\gtrsim} \mathbb{E}\left(\varphi(S_n^{\theta\xi}) \mid S_n^{\theta\xi} > x\right). \end{aligned} \tag{7.13}$$

Pagal min-max nelygybę (2.43) *asimptotinius rėžius* (7.13) *galima pilnai išreikšti per sąlyginius vidurkius, nors tokiu būdu rėžiai išeina ne tokie tikslūs:*

$$\begin{aligned} &\min_{1\leqslant k\leqslant n} \left\{ L_{F_{\xi_k}}^2 \mathbb{E}\left(\varphi(\theta_k\xi_k) \mid \theta_k\xi_k > x\right) \right\} \underset{x\to\infty}{\lesssim} \mathbb{E}\left(\varphi(S_n^{\theta\xi}) \mid S_n^{\theta\xi} > x\right), \\ &\max_{1\leqslant k\leqslant n} \left\{ \frac{1}{L_{F_{\xi_k}}^2} \mathbb{E}\left(\varphi(\theta_k\xi_k) \mid \theta_k\xi_k > x\right) \right\} \underset{x\to\infty}{\gtrsim} \mathbb{E}\left(\varphi(S_n^{\theta\xi}) \mid S_n^{\theta\xi} > x\right). \end{aligned}$$

7.5 Haezendonck-Goovaerts rizikos mato asimptotika

Nepaisant to, kad Haezendonck-Goovaerts rizikos matas yra suderintas ir, kaip ES rizikos mato apibendrinimas, suteikia daugiau lankstumo vertinant riziką, bendruoju atveju jis negali būti išreikštas analitiškai [108], nes neegzistuoja išreikštinis (7.9) sprendinys. Visgi, tam tikrais atvejais analitinės formulės gali būti išvestos. Vienas tokių atvejų, kai normalizuota Young funkcija yra laipsninė, t. y. $\varphi(t) = t^{\varkappa}$, $\varkappa \geqslant 1$ (žr. [108, 80]). Pavyzdžiui, Tang ir Yang [108] straipsnyje gavo žemiau 7.8 teoremoje pateikiamas HG mato išraiškas.

Teorema 7.8 (Žr. [108, 2.1 teoremą ir (1.3)]). *Tarkime, kad turime laipsninę normalizuotą Young funkciją $\varphi(t) = t^{\varkappa}$, $\varkappa \geqslant 1$.*

(i) *Jei $q \in (0,1)$, $\varkappa = 1$ ir $\mathbb{E}\xi^+ < \infty$, tuomet*

$$\mathrm{HG}_q(\xi) = F_\xi^{\leftarrow}(q) + \frac{\mathbb{E}\Big(\xi - F_\xi^{\leftarrow}(q)\Big)^+}{1-q}.$$

(ii) *Jei $q \in (0{,}1)$, $\varkappa > 1$, $\mathbb{P}(\xi = F_\xi^{\leftarrow}(1)) = 0$, ir $\mathbb{E}(\xi^+)^{\varkappa} < \infty$, tuomet*

$$\mathrm{HG}_q(\xi) = x + \left(\frac{\mathbb{E}\big((\xi - x)^+\big)^{\varkappa}}{1-q}\right)^{1/\varkappa}.$$

Čia $x = x(q) \in (-\infty, F_\xi^{\leftarrow}(1))$ yra vienintelis lygties

$$\frac{\Big(\mathbb{E}\big((\xi - x)^+\big)^{\varkappa-1}\Big)^{\varkappa}}{\Big(\mathbb{E}\big((\xi - x)^+\big)^{\varkappa}\Big)^{\varkappa-1}} = 1 - q$$

sprendinys.

Remiantis 7.8 teorema, šioje disertacijoje plėtojama HG mato asimptotika. Toliau pateikiamos dvi teoremos, kurios nusako HG mato asimptotinį elgesį atsitiktinėms sumoms su nuosaikiai kintančias uodegas turinčiais dėmenimis, atveju, kai normalizuota Young funkcija yra laipsninė, t. y. $\varphi(t) = t^{\varkappa}$, $\varkappa \geqslant 1$. Gauti rezultatai rodo, kad kai pasikliovimo lygmuo q yra artimas 1, HG rizikos matą galima asimptotiškai aproksimuoti išraiška, kuri leidžia išvengti galimai priklausomų a.d.s sumos skirstinio vertinimo. Pirmoji iš teoremų nusako HG rizikos mato asimptotinę išraišką nesvorinėms sumoms S_n^{ξ}. Svarbu atkreipti dėmesį, kad tokiu atveju, rizikos mato asimptotika priklauso tik nuo nuosaikiai kintančias skirstinio uodegas turinčių dėmenų – jei atsitiktinė suma taip pat sudaryta iš lengvesnes uodegas turinčių a.d.s, jie nedaro įtakos HG mato asimptotiniam elgesiui.

Teorema 7.9 (Žr. [98, 3 teorema]). *Tarkime, kad turime laipsninę normalizuotą Young funkciją $\varphi(t) = t^{\varkappa}$, $\varkappa \geqslant 1$, ir realiųjų* pQAI *a.d.s rinkinį $\{\xi_1, \ldots, \xi_n\}$.*

(i) *Tegu $\varkappa = 1$, $\max_{1\leqslant k\leqslant n} \mathbb{E}(\xi_k^+) < \infty$, $F_{\xi_L} \in \mathscr{C}$ indeksui $L \in \{1, \ldots, n\}$, o atvejais, kai $k \neq L$, yra teisinga $F_{\xi_k} \in \mathscr{C}$ arba*

$\mathbb{P}(|\xi_k| > x) = o\big(\overline{F}_{\xi_L}(x)\big)$. *Jei* $F_{\xi_k} \in \mathscr{P}_{\mathscr{D}}$ *visiems* $k \in \mathcal{I}_n = \{k \in \{1,\ldots,n\} : F_{\xi_k} \in \mathscr{C}\}$, *tuomet*

$$\mathrm{HG}_q\Big(S_n^{\xi}\Big) \underset{q\uparrow 1}{\sim} H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+.$$

Čia $H_n^{\leftarrow}$ *yra pasiskirstymo funkcijos*

$$H_n := \max\Big\{0, 1 - \sum_{k\in\mathcal{I}_n}\overline{F}_{\xi_k}\Big\}$$

kvantilių funkcija.

(ii) *Tegu* $\varkappa \geqslant 2$, $\max_{1\leqslant k\leqslant n}\mathbb{E}\big(\xi_k^+\big)^{\varkappa} < \infty$, $F_{\xi_L} \in \mathscr{C}$ *indeksui* $L \in \{1,\ldots,n\}$, *o atvejais, kai* $k \neq L$, *yra teisinga* $F_{\xi_k} \in \mathscr{C}$ *arba* $\mathbb{P}(|\xi_k| > x) = o\big(\overline{F}_{\xi_L}(x)\big)$. *Taip pat tarkime, kad kažkokiam* $y > 1$

$$\left(\max_{k\in\mathcal{I}_n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(yx)}{\overline{F}_{\xi_k}(x)}\right)^{\varkappa}\left(\max_{k\in\mathcal{I}_n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_k}(yx)}\right)^{\varkappa-1} < 1.$$

Čia $\mathcal{I}_n = \big\{k \in \{1,\ldots,n\} : F_{\xi_k} \in \mathscr{C}\big\}$. *Tuomet*

$$\mathrm{HG}_q\Big(S_n^{\xi}\Big) \underset{q\uparrow 1}{\sim} \widehat{x}(q) + \left(\frac{1}{1-q}\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa}.$$

Čia $\widehat{x} = \widehat{x}(q)$ *yra vienintelis lygties*

$$\frac{\left(\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}\big)^+\Big)^{\varkappa-1}\right)^{\varkappa}}{\left(\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}\big)^+\Big)^{\varkappa}\right)^{\varkappa-1}} = 1 - q$$

sprendinys.

Žemiau pateikiama teorema yra 7.9 teoremos apibendrinimas atsitiktinėms svorinėms sumoms. Šį kartą daroma stipresnė prielaida, jog visi sumos dėmenys priklauso klasei $\mathscr{C}$.

Teorema 7.10 (Žr. [98, 3 teorema]). *Tarkime, kad turime laipsninę normalizuotą Young funkciją* $\varphi(t) = t^{\varkappa}$, $\varkappa \geqslant 1$, *realiųjų* pQAI *a.d.s rinkinį* $\{\xi_1,\ldots,\xi_n\}$, *tokį, kad* $F_{\xi_k} \in \mathscr{C}$ *visiems* $k = 1,\ldots,n$, *ir nebūtinai nepriklausomų, neneigiamų ir neišsigimusių nulyje a.d.s rinki-*

nį $\{\theta_1,\ldots,\theta_n\}$. *Taip pat tarkime, kad a.d.s rinkiniai* $\{\xi_1,\ldots,\xi_n\}$ *ir* $\{\theta_1,\ldots,\theta_n\}$ *yra nepriklausomi, o* $\max_{1\leqslant k\leqslant n}\{\mathbb{E}\theta_k^p\}$ *yra baigtinis kažkokiam* $p>\max_{1\leqslant k\leqslant n}\{J^+_{F_{\xi_k}}\}$.

(i) *Jei* $\varkappa=1$, $\max\limits_{1\leqslant k\leqslant n}\mathbb{E}(\xi_k^+)<\infty$, *o* $F_{\xi_k}\in\mathscr{P}_{\mathscr{D}}$ *visiems* $k\in\{1,\ldots,n\}$, *tuomet*

$$\mathrm{HG}_q\Big(S_n^{\theta\xi}\Big)\underset{q\uparrow 1}{\sim} H_n^{*\leftarrow}(q)+\frac{1}{1-q}\sum_{k=1}^{n}\mathbb{E}\Big(\theta_k\xi_k-H_n^{*\leftarrow}(q)\Big)^+.$$

Čia $H_n^{*\leftarrow}$ *yra pasiskirstymo funkcijos*

$$H_n^*:=\max\Big\{0,1-\sum_{k=1}^{n}\overline{F}_{\theta_k\xi_k}\Big\}$$

kvantilių funkcija.

(ii) *Tegu* $\varkappa\geqslant 2$, $\max\limits_{1\leqslant k\leqslant n}\mathbb{E}(\xi_k^+)^{\varkappa}<\infty$ *ir*

$$\left(\max_{1\leqslant k\leqslant n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(yx)}{\overline{F}_{\xi_k}(x)}\right)^{\varkappa}\left(\max_{1\leqslant k\leqslant n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_k}(yx)}\right)^{\varkappa-1}<1$$

kažkokiam $y>1$. *Tuomet*

$$\mathrm{HG}_q\Big(S_n^{\theta\xi}\Big)\underset{q\uparrow 1}{\sim} x^*(q)+\left(\frac{1}{1-q}\sum_{k=1}^{n}\mathbb{E}\Big(\big(\theta_k\xi_k-x^*(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa}.$$

Čia $x^*=x^*(q)$ *yra vienintelis lygties*

$$\frac{\left(\sum_{k=1}^{n}\mathbb{E}\Big(\big(\theta_k\xi_k-x^*\big)^+\Big)^{\varkappa-1}\right)^{\varkappa}}{\left(\sum_{k=1}^{n}\mathbb{E}\Big(\big(\theta_k\xi_k-x^*\big)^+\Big)^{\varkappa}\right)^{\varkappa-1}}=1-q$$

sprendinys.

Svarbu atkreipti dėmesį, kad 7.9 ir 7.10 teoremos neįtraukia atvejo $\varkappa\in(1,2)$. Šio atvejo analizė paliekama tolesniems tyrimams.

7.6 Išvados

Šioje disertacijoje buvo analizuojamos trijų tipų sunkiauodegių skirstinių transformacijos: nepriklausomų nulinio vidurkio normaliųjų a.d.s sandaugos, kurios esant pakankamam daugiklių skaičiui tampa sunkiauodegės, apibendrinti transformuotų atsitiktinių sumų su sunkiauodegiais dėmenimis momentai ir Haezendonck-Goovaerts rizikos matas sunkiauodegių a.d.s sumoms.

Pirmojoje disertacijos dalyje buvo nagrinėjamas nepriklausomų normaliųjų nulinio vidurkio a.d.s sandaugų skirstinio u.f. asimptotinis elgesys. Pagrindiniai rezultatai, pateikti 7.3 teoremoje ir 7.2 išvadoje, apibrėžia asimptotines formules su pirmos eilės liekamuoju nariu. Jos patikslina išraiškas, gautas [6] straipsnyje.

Antrojoje dalyje buvo gautos transformuotų atsitiktinių sumų, sudarytų iš galimai priklausomų, dominuojančiai kintančias uodegas turinčių a.d.s apibendrintų momentų asimptotinės formulės. Išraiškos pateiktos 7.6 ir 7.7 teoremose leidžia minėtus apibendrintus momentus asimptotiškai aprėžti atskirų transformuotų sumos dėmenų apibendrintų momentų su koreguojančiomis konstantomis sumomis. Pagrindiniai disertacijos rezultatai apibendrina ir patikslina ankstesnius kitų autorių rezultatus. Pirma, patikslinus koreguojančias konstantas ir jas įrašius po sumos ženklu gaunami tikslesni asimptotiniai rėžiai pagal pQAI struktūrą priklausomiems a.d.s. Antra, vietoje laipsninių funkcijų šeimos, teoremos apibrėžiamos platesnei transformacijų klasei. Galiausiai, atsitiktinių svorinių sumų atveju, aprėžtumo reikalavimas atsitiktiniams svoriams yra pakeičiamas mažiau ribojančia baigtinių momentų sąlyga.

Trečiojoje disertacijos dalyje buvo nagrinėjamos HG rizikos mato asimptotinės savybės atsitiktinei sumai su nuosaikiai kintančias uodegas turinčiais dėmenimis, kurių tarpusavio priklausomybę nusako minėta pQAI struktūra. Pagrindiniai rezultatai suformuluoti 7.9 ir 7.10 teoremose leidžia HG rizikos matą asimptotiškai vertinti specifiniu funkcionalu, struktūriškai primenančiu gautąjį Tang ir Yang [108] ir priklausančiu nuo sumos dėmenų marginaliųjų skirstinių. Norint įvertinti HG rizikos matą tokiu būdu nereikia gauti atsitiktinės sumos skirstinio išraiškos. Be to, 7.9 teorema teigia, jog nesvorinių sumų S_n^ξ atveju, kai kartu su klasei $\mathscr{C}$ priklausančiais dėmenimis suma sudaryta iš lengvesnes uodegas turinčių a.d.s, pakanka įvertinti nuosaikiai kintančius sumos dėmenų skirstinius.

Pabaigoje pateikiamos kryptys tolesniems tyrimams. Pirmiausia, asimptotiniai rezultatai normaliųjų a.d.s sandaugoms galėtų būti apibendrinti atvejui, kai normalieji daugikliai nebūtinai yra nulinio vidurkio. Be to, asimptotines formules 7.3 teoremoje ir 7.2 išvadoje būtų galima patikslinti įtraukiant papildomus aukštesnės eilės liekamuosius narius.

Antra, rezultatus baigtinėms sunkiauodegių a.d.s sumoms aprašomus 7.4 ir 7.5 skyreliuose būtų galima apibendrinti begalinėms sumoms

$$S_\infty^{\theta\xi} := \sum_{k=1}^{\infty} \theta_k \xi_k \qquad \text{ir} \qquad S_\infty^{\xi} := \sum_{k=1}^{\infty} \xi_k,$$

ar sumos dėmenų maksimumui arba supremumui:

$$\begin{aligned} M_n^{\theta\xi} &:= \max\{\theta_1\xi_1, \ldots, \theta_n\xi_n\}, \qquad & M_n^{\xi} &:= \max\{\xi_1, \ldots, \xi_n\}, \\ M_\infty^{\theta\xi} &:= \sup_{1\leqslant k<\infty} \{\theta_k\xi_k\}, & M_\infty^{\xi} &:= \sup_{1\leqslant k<\infty} \{\xi_k\}. \end{aligned} \tag{7.14}$$

Svarbu atkreipti dėmesį, kad (7.14) turi tiesioginę prasmę diskretaus laiko rizikos modelyje, vertinant (baigtinio laiko) bankroto tikimybę (daugiau žr. [71, 1 skyrius]):

$$\begin{aligned} \psi(x,n) &:= \mathbb{P}\left(S_k^{\theta\xi} > x \quad \text{kažkokiam} \quad 1 \leqslant k \leqslant n\right) = \mathbb{P}\left(M_n^{\theta\xi} > x\right) \quad \text{arba} \\ \psi(x) &:= \mathbb{P}\left(S_k^{\theta\xi} > x \quad \text{kažkokiam} \quad 1 \leqslant k < \infty\right) = \mathbb{P}\left(M_\infty^{\theta\xi} > x\right). \end{aligned}$$

Čia x atitinka pradinį draudimo kompanijos kapitalą, o $S_k^{\theta\xi}$ atspindi per pirmus k periodų sukauptų nuostolių dabartinę vertę. Taip pat, transformuotų atsitiktinių sumų apibendrintų momentų asimptotiką galima būtų nagrinėti platesnėje sunkiauodegių skirstinių klasėje arba naudojant mažiau nei pQAI ribojančią priklausomybės struktūrą.

Trečia, svarbu atkreipti dėmesį, kad HG rizikos mato asimptotiniai rezultatai pateikti 7.9 ir 7.10 teoremose neįtraukia atvejo, kai laipsninės Young funkcijos laipsnis $\varkappa \in (1,2)$. Taip pat, panašius asimptotinius rezultatus būtų galima praplėsti skirstiniams priklausantiems klasei $\mathscr{D}$. Tokiu atveju, tikėtina, asimptotiškai ekvivalenčią išraišką reikėtų pakeisti asimptotiniais rėžiais arba, norint išlaikyti tikslią asimptotiką, pridėti papildomus apribojimus. Galiausiai, būtų galima nagrinėti platesnį normalizuotų Young funkcijų poklasį, panašiai kaip [109].

7.7 Rezultatų sklaida

7.7.1 Publikacijos

Disertacijoje pateikti tyrimų rezultatai buvo publikuoti žemiau pateikiamuose 4-iuose moksliniuose straipsniuose:

- M. Dirma, S. Paukštys ir J. Šiaulys. Tails of the moments for sums with dominatedly varying random summands. *Mathematics*, 9(8):824, 2021.
- R. Leipus, J. Šiaulys, M. Dirma ir R. Zovė. On the distribution-tail behaviour of the product of normal random variables. *Journal of Inequalities and Applications*, 2023(1):32, 2023.
- M. Dirma, N. Nakliuda ir J. Šiaulys. Generalized moments of sums with heavy-tailed random summands. *Lithuanian Mathematical Journal*, 63(3):254-271, 2023.
- J. Šiaulys, M. Dirma, N. Nakliuda ir L. Zanardelli. Asymptotic Formulas for the Haezendonck–Goovaerts Risk Measure of Sums with Consistently Varying Increments. *Axioms*, 15(1):20, 2026.

Visi straipsniai buvo išleisti ISI Web of Science duomenų bazėje indeksuojamuose žurnaluose.

7.7.2 Konferencijos ir moksliniai seminarai

Gauti tyrimų rezultatai buvo pristatyti žemiau išvardytose konferencijose ir moksliniuose seminaruose:

- M. Dirma ir N. Nakliuda, *Asimptotinės formulės sunkiauodegių skirstinių sumų apibendrintiems momentams.* Finansų ir draudimo matematikos seminaras, 2022 gruodžio 13 d., Vilnius.
- M. Dirma, *Asimptotinės formulės sunkiauodegių skirstinių sumų apibendrintiems momentams.* 11-asis Lietuvos jaunųjų matematikų susitikimas, 2022 gruodžio 30 d., Kaunas.
- M. Dirma, *Apibendrinti atsitiktinių sumų su sunkiauodegiais dėmenimis momentai.* 64-oji Lietuvos Matematikų Draugijos konferencija, 2023 birželio 21-22 d., Vilnius.

- M. Dirma, *Normaliųjų atsitiktinių dydžių sandaugų skirstinio uodegos asimptotinis elgesys*, 65-oji Lietuvos Matematikų Draugijos konferencija, 2024 birželio 27–28 d., Šiauliai.

- M. Dirma, *On the Distribution-Tail Behaviour of the Product of Normal Random Variables*, International Conference on Probability Theory and Number Theory 2024, 2024 rugsėjo 16–20 d., Palanga.

- M. Dirma ir N. Nakliuda, *Haezendonck-Goovaerts rizikos mato asimptotika skirstinių su nuosaikiai kintančiomis uodegomis sumoms*, 66-oji Lietuvos Matematikų Draugijos konferencija, 2025 birželio 26–27 d., Klaipėda.

- M. Dirma, *On the asymptotic behaviour of the Haezendonck-Goovaerts risk measure for sums with consistently varying increments*, International Vilnius Conference on Statistics and Its Applications, 2025 rugpjūčio 27–29 d., Vilnius.

7.8 Trumpos žinios apie autorių

Išsilavinimas:

- 2013–2015 m. Vilniaus licėjus, vidurinis išsilavinimas.

- 2015–2019 m. Vilniaus universitetas, Matematikos ir informatikos fakultetas, finansų ir draudimo matematikos bakalauras *(Cum Laude)*.

- 2019–2021 m. Vilniaus universitetas, Matematikos ir informatikos fakultetas, finansų ir draudimo matematikos magistras *(Cum Laude)*.

- 2021–2025 m. Vilniaus universitetas, Matematikos ir informatikos fakultetas, gamtos mokslų matematikos kryptis, doktorantūros studijos.

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