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Effective Variance Estimates of Additive Functions Defined on Random Combinatorial Objects

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VILNIAUS UNIVERSITETAS

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Efektyvūs dispersijų įverčiai adityvio-siomis funkcijoms, apibrėžtoms virš atsitiktinių kombinatorinių objektų

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Contents

Notation

$\mathbb{N}$	the set of natural numbers (positive integers)
$\mathbb{N}_0$	the set of natural numbers including zero
$\mathbb{N}_0^n$	the set of non-negative integer vectors of dimension n
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of positive real numbers
$\mathbb{R}^n$	the set of positive real number vectors of dimension n
$\bar{a}$	an arbitrary vector $\bar{a} \in \mathbb{R}^n$
$\mathbb{R}(\bar{a})$	the linear hull spanned by the vector $\bar{a} \in \mathbb{R}^n$ (or $\mathrm{span}\{\bar{a}\}$)
$\mathbb{C}$	the set of complex numbers
r.v.	random variable
$\mathbf{1}\{\cdot\}$	the indicator function
i, j, l, k, n, m, m_1 and m_2	some natural numbers
$O(\cdot)$	the big-O notation
$\ll$	the analog of $O(\cdot)$

Θ	any real value whose absolute value does not exceed 1
$\|\|\bar{z}\|\|$	the Euclidean norm of a vector $\bar{z} \in \mathbb{R}^n$
$\mathrm{Trace}(\cdot)$	the trace of a matrix
$\mathrm{Sp}(\cdot)$	the spectrum of a matrix
$\ell(\bar{k}) = 1k_1 + \cdots + nk_n$	the mapping $\ell : \mathbb{N}_0^n \to \mathbb{N}$
$(x)_n = x(x-1)\cdots(x-n+1)$	the falling factorial of $x \in \mathbb{R}$
$\mathbb{S}_n$	the symmetric group on n elements
σ	an arbitrary permutation $\sigma \in \mathbb{S}_n$
$\omega(\sigma) = \omega$	the number of cycles in $\sigma \in \mathbb{S}_n$
$\bar{\kappa}(\sigma) = \big(\kappa_1(\sigma), \ldots, \kappa_n(\sigma)\big)$	the cycle structure vector of $\sigma \in \mathbb{S}_n$
$\psi_r(u) = uP_{r-1}^{(1,0)}(1-2u)$	the Jacobi polynomials, $r \in \mathbb{N}$
$\mathcal{G}$	the set of weighted multisets
$\mathcal{G}_n$	the set of weighted multisets of weight n
ϱ	an arbitrary weighted multiset $\varrho \in \mathcal{G}_n$
$\pi(n)$	the number of initial weighted objects of weight n comprising the multisets in $\mathcal{G}$
$m(n)$	the number of multisets of weight n in $\mathcal{G}$
$\mathbb{F}_q$	the finite field of size q
$\mathbb{F}_q^*[t]$	the set of monic polynomials over a finite field $\mathbb{F}_q$
$\bar{k}(\varrho) = (k_1(\varrho), \ldots, k_n(\varrho))$	the component structure vector of a weighted multiset $\varrho \in \mathcal{G}_n$
$\mu(\cdot)$	the Möbius function

ν_n	the discrete uniform probabilistic measure on $\mathbb{S}_n$ or $\mathcal{G}_n$
$\mathbf{E}_n$, $\mathbf{C}ov_n$ and $\mathbf{V}_n$	the expectation, covariance and variance with respect to ν_n
$\mathbf{E}$ and $\mathbf{V}$	the expectation and variance with respect to an unspecified probability measure

A list of frequently used quadratic forms

$$B_n(\bar{a}) = \sum_{j\le n} a_j^2/j$$

$$D_n(\bar{a}) = \sum_{j=1}^n a_j^2 p_j(n), \quad p_j(n) = \sum_{1\le k\le n/j} \frac{(-1)^{k+1}}{j^k k!}$$

$$\Delta_{n1}(\bar{a}) = \sum_{\substack{i,j\le n\\ i+j>n}} a_i a_j p_i(n) p_j(n)$$

$$\Delta_{n2}(\bar{a}) = \sum_{\substack{i+j\le n\\ i\ne j}} a_i a_j [p_i(n)p_j(n) - p_{i,j}(n)],$$

$$p_{i,j}(n) = \sum_{\substack{l,k\ge 1\\ il+jk\le n}} \frac{(-1)^{l+k}}{i^l l! j^k k!}$$

$$\Delta_{n3}(\bar{a}) = \sum_{j\le n/2} a_j^2 p_j^2(n)$$

$$\mathbf{B}_n(\bar{a}) = \sum_{1\le jk\le n} a_j^2 \pi(j) k q^{-jk}$$

$$\mathbf{R}_n(\bar{a}) = \sum_{m\le n} m q^{-2m} \Big(\sum_{j|m} a_j \pi(j) \Big)^2$$

Chapter 1

Introduction

1.1 Research problem

Variance estimates for additive functions defined on random permutations and random weighted multisets.

1.2 Actuality

Permutations take a central role in combinatorics. They appear naturally in a plethora of problems that feature rearrangements of a finite set. In addition, permutations form the basis of the symmetric group, whose properties capture both algebraic and geometric symmetries. This and other connections make permutations indispensable in many mathematical fields.

One of the ways to investigate the properties of permutations is to analyze additive functions that are defined on them. These functions can be used to count permutations whose cycle lengths satisfy various specified conditions (see, e.g. [49]), or to approximate important mappings on the symmetric group having algebraic sense ([9], [51]). Furthermore, additive functions are present in some physical models, in particular, they comprise a part of Hamiltonians in the Bose gas theory (see [5] and the references therein). In addition, the properties of other combinatorial structures can also be encoded in an additive sense (see, e.g. [36]). For such functions, it is useful to obtain variance bounds, which are crucial in proving limit theorems [3], proving laws of large numbers [35], or defining functional spaces [34]. The variance bounds of additive functions are the main topic of the first part of this work.

The second part of this thesis concerns additive functions defined on weighted multisets. Main interest in it stems from the semigroup of monic polynomials over a finite field of a prime power size. An additive function counts the number of irreducible polynomials, which is of great interest in cryptography and coding theory, see [6, Chapter 2]. Furthermore, the degree of the splitting field can be approximated by additive functions (see, e.g. [7], [51], [50, Chapter 2], [42]). In this thesis, we will work with a slightly more general structure called multisets of necklaces, which encompasses monic polynomials over a finite field. It can be interpreted as words over a finite alphabet, up to cyclic equivalence [41]. Similarly to permutations, general properties of additive functions enable the proofs of more advanced results.

1.3 Aims and tasks

The main aim of this thesis is to investigate the asymptotics of the variance of additive functions that are defined on decomposable combinatorial structures. More specifically:

- Obtain variance estimates with asymptotic constants for strongly additive functions defined on random permutations with the best possible remainder terms.
- Obtain variance estimates with asymptotic constants for completely additive functions defined on random weighted multisets with the best possible remainder terms.

1.4 Methods

In this thesis, methods from enumerative and probabilistic combinatorics, number theory, linear algebra, and numerical analysis are used. Inspiration from number theory is drawn from the works of J. Kubilius [26] and J. Lee [27], where methods of covariance matrix analysis and orthogonal polynomial properties are used. Furthermore, the work of J.Klimavičius and E. Manstavičius [19] is crucial in estimating some quadratic forms that appear in the variance expression. Finally, the power method algorithm for finding the largest eigenvalue [44] is also used as inspiration for deriving asymptotic expressions for matrix eigenvalues.

1.5 Novelty

All the results described in this thesis are new. They extend the previously developed theory of additive functions defined on random combinatorial objects.

1.6 Approbation

Research results were disseminated at the following conferences:

- International Conference on Probability Theory and Number Theory, Palanga (Lithuania), 2024, *Variance of a Strongly Additive Function Defined on Random Permutations.*
- 2025 International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2025), Toronto (Canada), 2025, *On Covariance Matrix Related to Components of a Random Combinatorial Structure.*
- 66^{th} Conference of Lithuanian Mathematical Society, Klaipėda (Lithuania), 2025, *Keturiasdešimtmetės Kubiliaus nelygybės analogas keitiniams* (presented by the co-author E. Manstavičius).

1.7 Principal publications

- A. Karbonskis, E. Manstavičius, Effective bounds of the variance of statistics on multisets of necklaces, *Lietuvos matem. rink.*, 61(A):7–12, 2021.
- A. Karbonskis, E. Manstavičius, Variance of a strongly additive function defined on random permutations, *Lithuanian Mathematical Journal*, 61(3):1–13, 2024.

- A. Karbonskis, E. Manstavičius, Variance of statistics defined on random permutations, *Lithuanian Mathematical Journal*, 65(4):548–563, 2025.

1.8 Historical overview

This thesis examines the problem of estimating the variance of sums of dependent random variables that appear in combinatorics. The initial motivation is the success of the Turán–Kubilius inequality for the variance of additive number theoretic functions $f\colon \mathbb{N} \to \mathbb{R}$. By definition, such a function has the decomposition

$$f(m) = \sum_{p|m} f\left(p^{\alpha_p(m)}\right) \quad \text{if} \quad m = \prod_{p|m} p^{\alpha_p(m)},$$

where p is a prime number, $\alpha_p(m) \in \mathbb{N}$, and the notation $p|m$ stands for the expression "p divides m". If $f\left(p^k\right) = f(p)$ for every prime p and $k \in \mathbb{N}$, then f is called *a strongly additive* function. In this particular case, the inequality can be formulated as follows:

$$\frac{1}{n}\sum_{m=1}^{n}\left(\sum_{p|m} f(p) - \sum_{p\le n}\frac{f(p)}{p}\right)^2 =: K_n(f)\sum_{p\le n}\frac{f(p)^2}{p}, \quad K_n(f) \le C, \tag{1.1}$$

where $C > 0$ is an absolute constant and the values of the function $f(p)$ are arbitrary real numbers. The first inequality of this type, in the case when $f(p)$ values are bounded, was formulated and applied by P. Turán [46]. In his formulation, $K_n(f)$ was not an absolute constant since the right-hand side still involved the sum of the first moments. The form presented above was discovered by J. Kubilius [23]. A conjecture was posed that the bound $C = \frac{3}{2}$ is unimprovable for all $n \ge 2$ [24]. Moreover, J. Kubilius [26] proved that

$$\sup_{f\neq 0} K_n(f) = \frac{3}{2} + O\Big(\frac{1}{\log n}\Big).$$

Here and afterward, the constant in the symbol $O(\cdot)$ or in its analog $\ll$ will be absolute. J. Lee [27], [28] refined this result by deriving the inequality

$$K_n(f) \le \frac{3}{2} - \frac{C_1}{\log n},$$

for some $C_1 > 0$, provided that n is sufficiently large. For "small" values of n, the conjecture has not been verified computationally.

The Turán–Kubilius inequality is an effective estimate of the variance of sum of the random variables (r.vs) $f(p)\mathbf{1}\{\alpha_p(m) \geq 1\}$, $p \leq n$, when $m \leq n$ is sampled uniformly from the set $\mathbb{N}_n := \{1, \ldots, n\}$. Here and subsequently, $\mathbf{1}\{\cdot\}$ denotes the indicator function. For a fixed prime number p, the r.v. $\mathbf{1}\{\alpha_p(m) \geq 1\}$ converges in distribution to the Bernoulli r.v. with parameter $1/p$. The sum on the right-hand side of (1.1) is just the sum of their second moments. The appearance of the constant $3/2$ (not of 1) is the price we pay for the fact that the events $p_1|m$ and $p_2|m$ for the different primes $p_1, p_2 \leq n$ are dependent.

For a wider discussion concerning this inequality, we refer to P.D.T.A. Elliott's [8] and G. Tenenbaum's [45] books. The thesis [27] contains the earlier history of the problem; the mentioned important paper [26] is unfortunately missing in the references. Nowadays, the notion *Turán-Kubilius inequality* counts hundreds of references.

Combinatorial objects, such as permutations, polynomials over a finite field, or, more generally, assemblies and weighted multisets, are, much like natural numbers constructed from primes, also composed of corresponding simpler primitive structures (components). When selected randomly, the components exhibit dependencies analogous to the dependence of prime factors in a random natural number. Using this analogy with probabilistic number theory, a theory of additive functions defined on random structures has been developed. Specifically, we are interested in the theory of random permutations and random weighted multisets.

In particular, a permutation σ from the symmetric group $\mathbb{S}_n$ can be decomposed into a product of disjoint cycles

$$\sigma = \mathfrak{c}_1 \cdots \mathfrak{c}_w, \tag{1.2}$$

where $w = w(\sigma)$ is the number of cycles in σ. The alternative representation of σ is the labelled digraph G_σ with vertex set $V(\sigma) = \mathbb{N}_n$ and with the oriented cycles as components. A typical cycle $\mathfrak{c}$ of length j has the vertex set $\{k_1, \ldots, k_j\} \subset \mathbb{N}_n$ defined by

$$k_1 \xrightarrow{\sigma} k_2 \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} k_j \xrightarrow{\sigma} k_1.$$

To exploit the analogy with number theory, one deals with the mappings $h\colon \mathbb{S}_n \to \mathbb{R}$, whose values depend on the *cycle structure vector*

$$\bar{\kappa}(\sigma) := \big(\kappa_1(\sigma), \dots, \kappa_n(\sigma)\big),$$

where $\kappa_j(\sigma)$ is the number of cycles of length j in product (1.2). We say that h is *additive* if the following partition holds:

$$h(\sigma) = \sum_{j \le n} h_j(\kappa_j(\sigma)),$$

with some functions $h_j\colon \mathbb{N}_0 \to \mathbb{R}$ satisfying $h_j(0) = 0$ for each $j \le n$. If $h_j(k) = a_j \mathbf{1}\{k \ge 1\}$ with $a_j \in \mathbb{R}$, $j \le n$, we call the function h *strongly additive*. Moreover, if $h_j(k) = h_j(1)k$ for every $j \le n$ and $k \in \mathbb{N}$, the function h is called *completely additive*.

We now introduce the notation of random permutation theory. Let ν_n be the uniform probability measure on $\mathbb{S}_n$, $\mathbf{E}_n$ and $\mathbf{V}_n$ denote the expectation and the variance with respect to ν_n. Now the coordinates of the cycle vector taken with respect to ν_n become r.vs. $\kappa_j := \kappa_j(\sigma)$ for $j \le n$. Denote $\ell(\bar{s}) := 1s_1 + \cdots + ns_n$ for a vector $\bar{s} = (s_1, \dots, s_n) \in \mathbb{N}_0^n$ and observe that $\ell\big(\bar{\kappa}(\sigma)\big) = n$ for each $\sigma \in \mathbb{S}_n$. Now we mention the most important historical results on random permutations, as well as those very closely related to our topic.

If ξ_j, $j \le n$, are independent Poisson r.vs with the parameters $1/j$ defined on some probability space endowed with the probability measure P and $\bar{\xi} := (\xi_1, \dots, \xi_n)$, then

$$\nu_n(\bar{\kappa} = \bar{s}) = \mathbf{1}\{\ell(\bar{s}) = n\} \prod_{j \le n} \frac{1}{j^{s_j} s_j!} = P\big(\bar{\xi} = \bar{s} | \ell(\bar{\xi}) = n\big). \tag{1.3}$$

This so-called *conditioning relation* was first discovered by V.L. Goncharov [11], [12]. Relation (1.3) allows us to interpret many additive statistics defined on permutations as functionals of independent Poisson r.vs. conditioned on the total size. He also investigated the behavior of a completely additive function $\omega(\sigma)$, finding an asymptotic expression for the mean and variance and proving the appropriate CLT. This marked the beginning of the random permutation theory.

The next notable result was due to P. Erdös and P. Turán [9]. They investigated the group theoretical order of a randomly picked permutation $\sigma \in \mathbb{S}_n$. It was shown that the logarithm of it can be approximated by the additive function

$$\sum_{j \leq n} \kappa_j(\sigma) \log j.$$

The authors showed that if appropriately normalized, this function also follows a standard normal distribution. Currently, the best result on the convergence of the distribution for a random permutation σ is obtained by V. Zacharovas [51]. His work is based largely on the techniques of [30], which were developed for general sequences of completely additive functions.

A thorough probabilistic treatment of the structural cycle structure of random permutations was developed by R. Arratia and S. Tavaré [2]. It formed the basis for modern approaches to additive functions via Poisson approximation. This and other results on random permutations of the last century are described in [1].

E. Manstavičius investigated the problem of integral asymptotic laws of additive functions [29]. The aim is to find sufficient conditions such that $\nu_n(h_n(\sigma) - A(n) < x)$ converges weakly to a limit distribution law, where $h_n(\sigma)$ - a sequence of completely additive functions and $A(n)$ - some normalizing sequence, close to $\mathbf{E}_n h_n(\sigma)$. Furthermore, in [35], the author finds necessary and sufficient conditions for the weak law of large numbers.

We also mention a couple of the most recently established results concerning variance estimates for additive functions defined on random permutations. J. Klimavičius and E. Manstavičius [19] gives the sharp estimate of $\mathbf{V}_n h$ for a completely additive function $h(\sigma)$ in terms of

$$B_n(\bar{a}) := \sum_{j \leq n} \frac{a_j^2}{j}, \ \bar{a} \in \mathbb{R}^n,$$

which equals to the sum of variances of independent r.vs $a_j \xi_j, \ j \leq n$. Then

$$\sup \Big\{ \mathbf{V}_n h / B_n(\bar{a}) : \ \bar{a} \neq 0, \, n \geq 2 \Big\} = \frac{3}{2}.$$

The proof was built upon searching for the minimal eigenvalue of the matrix, which appears in the quadratic form $\mathbf{V}_n h$. The corresponding eigenvectors were expressed in terms of the Hahn polynomials [37]. That was a notable difference from the number-theoretic technique, which had involved the Jacobi polynomials [26].

This result was further generalized by Ž. Baronėnas, E. Manstavičius, and P. Šapokaitė [4]. Instead of sampling all permutations uniformly, Ewens probability measure $\nu_{n,\theta}(\{\sigma\}) = \theta^{\omega(\sigma)}/(\theta)_n$ was used. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial. Let $\Theta(m) = (\theta)_m/m!$, $m \in \mathbb{N}$. Then the variance of a completely additive function $h(\sigma)$ with respect to Ewens probability measure has the expression

$$\begin{aligned}\mathbf{V}_{n,\theta}h &= \theta \sum_{j\le n} \frac{a_j^2}{j}\frac{\Theta(n-j)}{\Theta(n)} \\ &\quad + \theta^2 \sum_{i+j\le n} \frac{a_i a_j}{ij}\frac{\Theta(n-i-j)}{\Theta(n)} - \theta^2 \sum_{j\le n}\Bigg(\sum_{j\le n}\frac{a_j}{j}\frac{\Theta(n-j)}{\Theta(n)}\Bigg)^2.\end{aligned}$$

In this case

$$\sup\left\{\frac{\mathbf{V}_{n,\theta}h}{B_{n,\theta}(\bar a)} \,:\, \bar a \ne 0,\, n\ge 2\right\} = \frac{\theta+2}{\theta+1},$$

where

$$B_{n,\theta}(\bar a) = \theta\sum_{j\le n}\frac{a_j^2}{j}\frac{\Theta(n-j)}{\Theta(n)}.$$

By choosing $\theta = 1$ we arrive at the previous estimate in [19]. So far, the best O-type variance estimates for general additive functions with respect to Ewens probability measure were obtained by E. Manstavičius and V. Stepas [39].

The power moments of additive functions on uniform random permutations were previously investigated by E. Manstavičius [33]. Related power moment estimates with respect to the Ewens probability measure were later developed in [40] by E. Manstavičius and V. Stepas. In both works, the authors obtained moment bounds for general additive functions by directly analyzing the appearing cumbersome expressions.

To get the O-type estimates, there exists an indirect approach based upon appropriate moment inequalities for sums of independent r.vs and Manstavičius' [31] relation

$$\nu_n\Bigg(\Bigg|\sum_{j\le n} h_j(\varkappa_j(\sigma)) - a\Bigg| \ge 3u\Bigg) \le 32\mathrm{e}^2 P\Bigg(\Bigg|\sum_{j\le n} h_j(\xi_j) - a\Bigg| \ge u\Bigg).$$

where $h_j : \mathbb{N}_0 \to \mathbb{R}$ and $h_j(0) = 0$ if $j \le n$, $a \in \mathbb{R}$, $u \in \mathbb{R}_+$ are arbitrary. Later, it was observed [32] that the factor 3 of u can not be substituted even by $7/5$; therefore, any refinement of the constant $32\mathrm{e}^2$ will not lead to the discovery of the optimal constants in variance estimates.

As mentioned, ideas from number theory regarding additive functions can be applied not only to permutations but also to other classes of decomposable structures. In this thesis, we are also interested in the theory of additive arithmetic semigroups as formulated by J. Knopfmacher [21]. Classes of weighted multisets and arithmetic semigroups are practically the same structures that differ only in interpretation (see [20], [10, Chapter II]). Accordingly, they will be treated as identical in this thesis.

Many authors have contributed to the probabilistic theory of additive functions defined on arithmetic semigroups ([15], [14], [13], [38], [43], [22], [20]). In our case, we are specifically interested in variance estimates for the aforementioned functions. W.-B. Zhang in [52] investigated the existence of limit distributions for additive functions defined on the elements of arithmetic semigroups. In [52] variance estimates analogous to the Turán-Kubilius inequality are also provided. The work of S. Wehmeier is also a notable contribution to the topic. In the dissertation [47, Chapter III] and the article [48], the author obtained variance estimates for strongly additive functions, which we cite in their original notation.

Let $(G, \cdot)$ be the free commutative monoid over a nonempty set P of "primes". Define a degree function $\partial : G \to \mathbb{N}_0$, satisfying $\partial(ab) = \partial(a)+\partial(b)$ for all $a, b \in G$. Denote $G_n = \{a \in G : \partial(a) = n\}$ and assume $G(n) = |G_n|$ to be finite. Let $f : G \to \mathbb{R}$, such that $f(a) = \sum_{p|a} f(p)$, $a \in G$. Suppose $G(n) = (A + O((\log n)^{-1})q^n$, where $q > 1$ and $A > 0$. In addition, assume the Chebyshev condition holds

$$P(k) \le C_G \max_{\substack{n\in\mathbb{N} \\ G(n-k)>0}} \frac{G(n)}{kG(n-k)},$$

for all $k \in \mathbb{N}$ and some positive C_G, where $P(k) = |\{p \in P : \partial(p) = k\}|$. Denote

$$A(n) = \frac{1}{G(n)} \sum_{\partial(p)\le n} f(p)G(n - \partial(p)),$$

$$B^2(n) = \frac{1}{G(n)} \sum_{\partial(p)\le n} f(p)^2 G(n-\partial(p)),$$

then

$$\frac{1}{G(n)} \sum_{a\in G_n} (f(a) - A(n))^2 = O(B^2(n)).$$

So far, this is the most general O-type variance estimate for the class of strongly additive functions defined on arithmetic semigroups.

1.9 Main results

In this section, we present the main results of this thesis. Proofs are detailed in subsequent Chapters.

1.9.1 Strongly additive functions defined on random permutations

In this section, the results for strongly additive functions defined on permutations are presented.

Henceforth, we will use a probabilistic interpretation of a strongly additive function

$$h(\sigma) = \sum_{j\le n} a_j \varkappa_j, \quad \varkappa_j = \varkappa_j(\sigma) := \mathbf{1}\{\kappa_j(\sigma) \ge 1\}, \quad j \le n.$$

In contrast to $\kappa_j(\sigma)$, $j \le n$, that appear in the decomposition of completely additive functions, the Bernoulli r.vs $\varkappa_j$ attain value 1 with probabilities

$$p_j(n) := \nu_n(\kappa_j \ge 1) = \mathbf{E}_n \varkappa_j = \sum_{1\le k\le n/j} \frac{(-1)^{k+1}}{j^k k!}$$

and are always pairwise correlated. Indeed, as shown in [17], we have

$$\begin{aligned}
\mathbf{C}ov_n(\varkappa_i, \varkappa_j) &= \nu_n(\varkappa_i = 1, \varkappa_j = 1) - \nu_n(\varkappa_i = 1)\nu_n(\varkappa_j = 1) \\
&= \sum_{\substack{l,k\ge 1 \\ il+jk\le n}} \frac{(-1)^{l+k}}{i^l l! j^k k!} - \sum_{1\le l\le n/i} \frac{(-1)^{l+1}}{i^l l!} \cdot \sum_{1\le k\le n/j} \frac{(-1)^{k+1}}{j^k k!} \\
&=: p_{i,j}(n) - p_i(n)p_j(n), \quad i \ne j.
\end{aligned}$$

This covariance expression is more complex than in the case of completely additive functions. In that case, a much simpler expression is present

$$\mathbf{C}ov_n(\kappa_i, \kappa_j) = \frac{\mathbf{1}\{i = j\}}{j} - \frac{\mathbf{1}\{i + j > n\}}{ij}.$$

We obtain the following initial expression of the variance

$$\begin{aligned}
\mathbf{V}_n h =: V_n(\bar{a}) &= \frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} h^2(\sigma) - \Big(\sum_{j \le n} a_j p_j(n) \Big)^2 \\
&= \sum_{j=1}^{n} a_j^2 p_j(n) - \sum_{\substack{i,j \le n \\ i+j>n}} a_i a_j p_i(n) p_j(n) \\
&\quad - \sum_{\substack{i+j \le n \\ i \ne j}} a_i a_j [p_i(n)p_j(n) - p_{i,j}(n)] - \sum_{j \le n/2} a_j^2 p_j^2(n) \\
&=: D_n(\bar{a}) - \Delta_{n1}(\bar{a}) - \Delta_{n2}(\bar{a}) - \Delta_{n3}(\bar{a}). \qquad (1.4)
\end{aligned}$$

The main goal is to obtain the upper bound for $\mathbf{V}_n h$ by using the first term $D_n(\bar{a})$, which is close to $\sum_{j \le n} \mathbf{V}_n(a_j \varkappa_j)$. Indeed,

$$B_n(\bar{a})/4 \le D_n(\bar{a})/2 \le \sum_{j \le n} \mathbf{V}_n(a_j \varkappa_j) \le D_n(\bar{a}) \le B_n(\bar{a}).$$

It is easy to show that omitting the last non-positive quadratic form introduces a negligible error

$$D_n(\bar{a}) - \Delta_{n3}(\bar{a}) - \sum_{j \le n} \mathbf{V}_n(a_j \varkappa_j) = \sum_{n/2 < j \le n} a_j^2 p_j^2(n) \le \frac{2}{n} D_n(\bar{a}).$$

The remainder is successfully estimated with the exact asymptotic constant. In addition to the upper bound, a lower bound is also obtained.

Theorem 1.1 ([17])**.** *Let $\bar{a} \in \mathbb{R}^n$ be arbitrary and $B_n(\bar{a})$ be as defined in the previous section. If*

$$h := h(\sigma) := \sum_{j \le n} a_j \varkappa_j, \quad \sigma \in \mathbb{S}_n,$$

then

$$\mathbf{V}_n h \le \Big(\frac{3}{2} + R(n) \Big) D_n(\bar{a}) \le \Big(\frac{3}{2} + R(n) \Big) B_n(\bar{a}), \qquad (1.5)$$

where $R(n) = O(n^{-1/3})$ with an absolute constant in the estimate.

Furthermore, if $\bar{a}$ satisfies $\sum_{j\le n} ja_jp_j(n) = 0$, then

$$\mathbf{V}_n h \ge \frac{2}{3} B_n(\bar{a})\big(1 + R(n)\big). \tag{1.6}$$

Furthermore, constants $3/2$ and $2/3$ are optimal in (1.5), (1.6) i.e., there exist some vectors $\bar{a}_2$ and $\bar{a}_3$, such that equalities are achieved. As seen from (1.4), the proof reduces to analysis of the involved quadratic forms. It is easy to estimate Δ_{n1} by applying inequalities derived in [19]. In contrast, the effect of Δ_{n2} was estimated by calculating the trace of the underlying matrix. We refined this estimate further, improving the asymptotic rate from $O(n^{-1/3})$ to a sharp $O(n^{-1})$ by using a more sophisticated method.

Theorem 1.2 ([18])**.** *Assume the notation of Theorem 1.1. Then the estimates in (1.5) hold with $R(n) = O(n^{-1})$.*

The proof utilizes the idea of the Power method algorithm for finding the largest eigenvalue of a matrix. By simulating two iterations of the algorithm, we obtained a better asymptotic rate. Nevertheless, the refinement of Theorem 1.2 is not possible without the analysis of the quadratic form Δ_{n2}, which was performed in proving Theorem 1.1.

We also include an alternative formulation for Theorem 1.2. After substitution $a_j = \sqrt{j} b_j$, $j \le n$, in the matrix form, the simplified inequality can be written as

$$||\bar{b}\Gamma_n||^2 = (\bar{b}\Gamma_n)(\Gamma_n'\bar{b}') \le \big(3/2 + O(n^{-1})\big)n!||\bar{b}||^2, \tag{1.7}$$

for all $\bar{b} \in \mathbb{R}^n$. Here $'$ stands for transposition and $\Gamma_n = ((\gamma_{j\sigma}))$ is the $n \times n!$ matrix with entries

$$\gamma_{j\sigma} = \sqrt{j}\big(\varkappa_j(\sigma) - 1/j\big), \quad j \le n, \ \sigma \in \mathbb{S}_n.$$

We see that (1.7) is just an effective estimate of the maximal eigenvalue of $\Gamma_n\Gamma_n'$, which is the same as for the matrix $\Gamma_n'\Gamma_n$ of order $n! \times n!$ (see Lemma 3.3 on page 30 in [8]). Thus, we obtain the dual form of inequality (1.7).

Corollary 1.3. *For all vectors $\bar{d} \in \mathbb{R}^{n!}$ with the generic coordinate d_σ, $\sigma \in \mathbb{S}_n$, we have the following sharp estimate*

$$\sum_{j\le n} j \bigg(\sum_{\sigma\in\mathbb{S}_n} d_\sigma\big(\varkappa_j(\sigma) - 1/j\big)\bigg)^2 \le \big(3/2 + O(n^{-1})\big)n!||\bar{d}||^2.$$

Such dualization, highly developed and applied in number theory by P.D.T.A. Elliott [8] and his followers, still awaits an analogous theory for permutations.

1.9.2 Completely additive functions defined on random multisets

Let $(\mathcal{P}, ||\cdot||)$ be a set of initial weighted objects and

$$\pi(j) := \big|\{p \in \mathcal{P} :\ ||p|| = j\}\big| < \infty$$

for every $j = 1, 2, \ldots$ Examine the set $\mathcal{G}$ with the extended weight function $||\cdot||$ of multisets comprised of $p \in \mathcal{P}$. Namely, $\varrho \in \mathcal{G}$ if $\varrho = \{p_1, \ldots, p_r\}$ and $||\varrho|| = ||p_1|| + \cdots + ||p_r||$ including the empty multiset $\emptyset$ of weight 0. Then

$$m(n) := |\mathcal{G}_n| := \big|\{\varrho \in \mathcal{G} :\ ||\varrho|| = n\}\big| = \sum_{\ell(\bar{k})=n} \prod_{j=1}^{n} \binom{\pi(j) + k_j - 1}{k_j},$$

where $\ell(\bar{k}) = 1k_1 + \cdots + nk_n$ if $\bar{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In this thesis, we deal with the multisets for which $m(n) = q^n$, where $q \geq 2$ is an arbitrary natural number. Usually, a weaker axiom $\mathcal{A}^{\#}$, but stronger than the one used by S. Wehmeier [47] is used during the investigation of multiset structures or, more specifically, arithmetical semigroups [22]. The axiom is defined as

$$m(n) = Aq^n + R_n, \quad A \in \mathbb{R}_+, \quad R_n = O(q^{cn}), \quad 0 \leq c < 1, \quad \text{as } n \to \infty.$$

We have chosen a stricter axiom for our analysis to simplify the variance expression for additive functions, which will be defined below.

If q is a prime power, then $\mathcal{G}$ may be interpreted as $\mathbb{F}_q^*[t]$, the set of monic polynomials over a finite field $\mathbb{F}_q$. Then $\mathcal{P}$ is the subset of irreducible polynomials. For an arbitrary $q \geq 2$, $q \in \mathbb{N}$, there exist combinatorial constructions, called *multisets of necklaces* satisfying $m(n) = q^n$ (see, [1, Example 2.12, p. 43]). For multisets, we have the following relations

$$\pi(n) = \frac{1}{n} \sum_{d|n} q^{n/d} \mu(d), \qquad q^n = \sum_{d|n} d\pi(d), \tag{1.8}$$

where in the summations, d runs over natural divisors of n and $\mu(d)$ stands for the Möbius function. The equalities are equivalent to the formal power series relation

$$\sum_{n=0}^{\infty} q^n x^n = \frac{1}{1-qx} = \prod_{j=1}^{\infty} (1-x^j)^{-\pi(j)}.$$

Take an $\varrho \in \mathcal{G}_n$ uniformly at random, that is, sample it with probability $\nu_n(\{\varrho\}) = q^{-n}$, $\quad n \in \mathbb{N}$ and $\nu_0(\{\emptyset\}) = 1$. The probability measure ν_n here is different from the one defined for permutations. If $k_j(\varrho) \geq 0$ is the number of elements p_i in $\varrho \in \mathcal{G}_n$ of weight j, then $\bar{k}(\varrho) = \big(k_1(\varrho), \ldots, k_n(\varrho)\big)$ is the structure vector of $\varrho \in \mathcal{G}_n$ satisfying $\ell(\bar{k}(\varrho)) = n$. Its distribution is

$$\nu_n\big(\bar{k}(\varrho) = \bar{s}\big) = \mathbf{1}\{\ell(\bar{s}) = n\} q^{-n} \prod_{j=1}^{n} \binom{\pi(j) + s_j - 1}{s_j},$$

where $\bar{s} = (s_1, \ldots, s_n) \in \mathbb{N}_0^n$ and $\mathbf{1}\{\cdot\}$ stands for the indicator function.

We are interested in the distribution with respect to ν_n of the completely additive functions

$$h(\bar{a}) := h(\bar{a}, \varrho) = a_1 k_1(\varrho) + \cdots + a_n k_n(\varrho), \quad \bar{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n.$$

The number of components in ϱ is such a function, namely, it equals $k_1(\varrho) + \cdots + k_n(\varrho)$. We refer to [1] for more sophisticated examples.

In this thesis, we examine the variance of $h(\bar{a})$, which is a sum of dependent random variables, as the relation $h(\bar{j}, \varrho) = \ell(\bar{k}(\varrho)) = n$, $\bar{j} := (1, 2, \ldots, n)$ for each $\varrho \in \mathcal{G}_n$ shows. We propose an approach to overcome technical obstacles stemming from dependence.

Subsequently, the expectations and variances with respect to ν_n will be denoted by $\mathbf{E}_n$ and $\mathbf{V}_n$ while, when the probability space $(\Omega, \mathcal{F}, P)$ is not specified, we will respectively use the notation $\mathbf{E}$ and $\mathbf{V}$. The summation indices i, j, l, k, m, m_1, and m_2 will be natural numbers. We will now present the results published in [16].

Theorem 1.4. *If $\bar{a} \in \mathbb{R}^n$ and $n \in \mathbb{N}_0$, then*

$$\mathbf{V}_n h(\bar{a}) = \sum_{1 \leq jk \leq n} a_j^2 \pi(j) k q^{-jk} - \sum_{\substack{il+jk>n \\ il \leq n, jk \leq n}} a_i a_j \pi(i) \pi(j) q^{-il-jk}. \tag{1.9}$$

It is known from [1] that, for a fixed j, the r.v. $k_j(\varrho)$ converges in distribution to the r.v. γ_j distributed according to the negative binomial law $NB(\pi(j), q^{-j})$. Let $\{\gamma_1, \gamma_2, \dots\}$ be mutually independent. Define the statistic $Y_n = a_1\gamma_1 + \cdots + a_n\gamma_n$. We shall see that the first sum on the right-hand side in (1.9) is close to $\mathbf{V}Y_n$. Estimating $\mathbf{V}_n h(\bar{a})$, we use the following quadratic forms:

$$\mathbf{B}_n(\bar{a}) := \sum_{1\le jk\le n} a_j^2 \pi(j) k q^{-jk}, \quad \mathbf{R}_n(\bar{a}) := \sum_{m\le n} m q^{-2m} \Big(\sum_{j|m} a_j \pi(j)\Big)^2.$$

Theorem 1.5. *If $n \ge 2$, then*

$$\mathbf{V}_n h(\bar{a}) \le \mathbf{B}_n(\bar{a}) + \frac{1}{2}\mathbf{R}_n(\bar{a}). \tag{1.10}$$

The inequality becomes an equality for

$$a_j = a_j^* := \frac{3}{\pi(j)} \sum_{d|j} d q^d \mu\Big(\frac{j}{d}\Big) - 2n - 1, \quad 1 \le j \le n. \tag{1.11}$$

Corollary 1.6. *If $n \ge 2$ and $\bar{a} \ne \bar{0}$, then*

$$\mathbf{V}_n h(\bar{a}) < \frac{3}{2}\mathbf{B}_n(\bar{a}) < \Big(\frac{3}{2} - \frac{q-1}{q} n q^{-n}\Big)\mathbf{V}Y_n. \tag{1.12}$$

The inequalities are trivial for functions proportional to $h(\bar{j}, \varrho) = n$ if $\varrho \in \mathcal{G}_n$, because of $\mathbf{V}_n h(\bar{j}) = 0$. A shift of $\bar{a}$ allows for further improvement of the estimate. Observe that either of $\mathbf{B}_n(\bar{a} - t\bar{j})$ and $\mathbf{R}_n(\bar{a} - t\bar{j})$ attain their minimums in $t \in \mathbb{R}$ at

$$t = t_{\bar{a}} := \frac{2}{(n+1)n} \sum_{m\le n} m q^{-m} \sum_{j|m} a_j \pi(j).$$

Theorem 1.7. *If $n \ge 3$, then*

$$\begin{aligned} \mathbf{B}_n(\bar{a} - t_{\bar{a}}\bar{j}) - \frac{1}{3}\mathbf{R}_n(\bar{a} - t_{\bar{a}}\bar{j}) &\le \mathbf{V}_n h(\bar{a}) \\ &\le \mathbf{B}_n(\bar{a} - t_{\bar{a}}\bar{j}) + \frac{1}{2}\mathbf{R}_n(\bar{a} - t_{\bar{a}}\bar{j}). \end{aligned} \tag{1.13}$$

Both inequalities in (1.13) *are optimal, as long as $\mathbf{R}_n$ is used in the approximation of the variance.*

Corollary 1.8. *If $n \ge 3$ and $\bar{a} \notin \mathbb{R}(\bar{j})$, then*

$$\frac{2}{3}\mathbf{B}_n(\bar{a} - t_{\bar{a}}\bar{j}) < \mathbf{V}_n h(\bar{a}) < \frac{3}{2}\mathbf{B}_n(\bar{a} - t_{\bar{a}}\bar{j}),$$

where $\mathbb{R}(\bar{j})$ is a linear hull of $\bar{j}$.

In addition to absolute estimates of $\mathbf{V}_n h(\bar{a})$, asymptotic expressions of supremum and infimum of $\frac{\mathbf{V}_n h(\bar{a})}{\mathbf{B}_n(\bar{a})}$ are obtained.

Theorem 1.9. *There exists a real $C > 0$, such that*

$$\frac{3}{2} - \frac{C}{n^2} \leq \sup_{\bar{a} \in \mathbb{R}^n \setminus \{\bar{0}\}} \frac{\mathbf{V}_n h(\bar{a})}{\mathbf{B}_n(\bar{a})} \leq \frac{3}{2},$$

when $n \geq 2$.

Theorem 1.10. *There exists a real $C > 0$, such that*

$$\frac{2}{3} \leq \inf_{\bar{a} \in \mathbb{R}^n \setminus \mathbb{R}(\bar{j})} \frac{\mathbf{V}_n h(\bar{a})}{\mathbf{B}_n(\bar{a})} \leq \frac{2}{3} + \frac{C}{n^2},$$

when $n \geq 3$.

Chapter 2

Strongly additive functions defined on permutations

In this chapter, we prove Theorem 1.1 stated in Section 1.9.1, using the previously introduced notation.

2.1 Auxiliary lemmata

First, we derive an exact expression of variance for an arbitrary strongly additive function. Let $\bar{\xi}^{(x)} = (\xi_1^{(x)}, \xi_2^{(x)}, \dots)$ be an infinite dimensional vector with coordinates that are independent r.vs. following the Poisson law with parameters $\mathbf{E}\xi_j^{(x)} = x^j/j,\ j \geq 1,\ 0 < x < 1$. Suppose that the structural vector $\bar{\kappa}(\sigma)$ is also expanded to an infinite dimension accordingly $\bar{\kappa}(\sigma) = (\kappa_1(\sigma), \kappa_2(\sigma), \dots, \kappa_n(\sigma), 0, 0, \dots) \in \mathbb{N}_0^\infty$.

Lemma 2.1. *Let $0 < x < 1$ and $\Psi\colon \mathbb{N}_0^\infty \to \mathbb{C}$ be a mapping such that the mean $\mathbf{E}\Psi\left(\bar{\xi}^{(x)}\right)$ exists and is finite. Then*

$$\mathbf{E}\Psi\left(\bar{\xi}^{(x)}\right) = (1-x)\sum_{n\geq 0} \mathbf{E}_n \Psi\left(\bar{\kappa}(\sigma)\right) x^n.$$

This follows from the conditioning relation for logarithmic combinatorial structures. For proof, see [10, Section II.4].

We formally introduce the notion of additive functions defined on random permutations.

Definition 2.1. *We will say that the function $h : \mathbb{S}_n \to \mathbb{R}$ is additive if the following partition is true*

$$h(\sigma) = \sum_{j \le n} h_j(\kappa_j(\sigma))$$

with some functions $h_j \colon \mathbb{N}_0 \to \mathbb{R}$ satisfying $h_j(0) = 0$ for each $j \le n$. Furthermore, if $h_j(k) = a_j \mathbf{1}\{k \ge 1\}$ with $a_j \in \mathbb{R}$, $j \le n$, we call the function h strongly additive.

Since we are picking permutations randomly, additive functions can be interpreted as r.vs. In this interpretation, strongly additive functions are comprised of a linear combination of Bernoulli r.vs. $\varkappa_j := \mathbf{1}\{\kappa_j(\sigma) \ge 1\}$. We will now calculate the parameter $\mathbf{E}_n \varkappa_j$ and $\mathbf{E}_n \varkappa_j \varkappa_i$, $i, j \le n, i \ne j$, which will be necessary when deriving the exact variance of strongly additive functions.

Lemma 2.2.

$$\mathbf{E}_n \varkappa_j = \sum_{1 \le k \le n/j} \frac{(-1)^{k+1}}{j^k k!} := p_j(n).$$

Also, if $i \ne j$, then

$$\mathbf{E}_n \varkappa_j \varkappa_i = \sum_{\substack{l,k \ge 1 \\ il + jk \le n}} \frac{(-1)^{l+k}}{i^l l! j^k k!} := p_{i,j}(n).$$

Proof. The main idea of the proof relies on Lemma 2.1.

1) Let $\Psi(\bar{s}) = \mathbf{1}\{s_j \ge 1\}$. Then

$$\mathbf{E}\Psi(\bar{\xi}^{(x)}) = \mathbf{E}\mathbf{1}\{\xi_j^{(x)} \ge 1\} = 1 - \mathbf{P}(\xi_j^{(x)} = 0) = \sum_{k \ge 1} \frac{(-1)^{k+1} x^{jk}}{j^k k!}.$$

Combining both sides of the equation, we get

$$\left(\sum_{k \ge 1} \frac{(-1)^{k+1} x^{jk}}{j^k k!} \right) \cdot \left(\sum_{n \ge 0} x^n \right) = \sum_{n \ge 1} x^n \mathbf{E}_n \varkappa_j.$$

It remains to compare the coefficients of x^n.

2) Let $\Psi(\bar{s}) = \mathbf{1}\{s_i s_j \geq 1\}$, $i \neq j$. Then using the independence of $\xi_i^{(x)}$ and $\xi_j^{(x)}$ we obtain

$$\mathbf{E}\Psi(\bar{\xi}^{(x)}) = \Big(\sum_{l\geq 1} \frac{(-1)^{l+1}x^{il}}{i^l l!}\Big) \cdot \Big(\sum_{k\geq 1} \frac{(-1)^{k+1}x^{jk}}{j^k k!}\Big) = \sum_{l,k\geq 1} \frac{(-1)^{l+k}x^{il+jk}}{i^l l! j^k k!}.$$

Combining both sides of the equation, we get

$$\Big(\sum_{l,k\geq 1} \frac{(-1)^{l+k}x^{il+jk}}{i^l l! j^k k!}\Big) \cdot \Big(\sum_{n\geq 0} x^n\Big) = \sum_{n\geq 1} x^n \mathbf{E}_n(\varkappa_i \varkappa_j).$$

By comparing the coefficients of x^n, we obtain the result. $\square$

Lemma 2.3. *Let h be a strongly additive function and $\bar{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Then its variance can be expressed in the following way*

$$\begin{aligned}
\mathbf{V}_n h &=: V_n(\bar{a}) \\
&= \sum_{j\leq n} a_j^2 p_j(n) - \sum_{\substack{i,j\leq n \\ i+j>n}} a_i a_j p_i(n) p_j(n) \\
&\quad - \sum_{\substack{i+j\leq n \\ i\neq j}} a_i a_j \big(p_i(n)p_j(n) - p_{i,j}(n)\big) - \sum_{j\leq n/2} a_j^2 p_j^2(n) \\
&=: D_n(\bar{a}) - \Delta_{n1}(\bar{a}) - \Delta_{n2}(\bar{a}) - \Delta_{n3}(\bar{a}).
\end{aligned}$$

Proof. The proof is a straightforward calculation using the variance formula $\mathbf{V}_n h = \mathbf{E}_n(h^2) - (\mathbf{E}_n h)^2$ and results of Lemma 2.2. $\square$

We also introduce another lemma that is crucial in estimating the quadratic form Δ_{n1}.

Lemma 2.4. *Let $b_m \in \mathbb{R}$, $1 \leq m \leq n$ and $n \geq 2$, then*

$$-\frac{1}{2} \sum_{1\leq m\leq n} m b_m^2 \leq \sum_{\substack{1\leq m_1, m_2\leq n \\ m_1+m_2>n}} b_{m_1} b_{m_2} \leq \sum_{1\leq m\leq n} m b_m^2. \tag{2.1}$$

Further, if $n \geq 3$ and $\sum_{m\leq n} m b_m = 0$, then

$$\sum_{\substack{1\leq m_1, m_2\leq n \\ m_1+m_2>n}} b_{m_1} b_{m_2} \leq \frac{1}{3} \sum_{1\leq m\leq n} m b_m^2. \tag{2.2}$$

In addition, both inequalities (2.1) and (2.2) are sharp, i.e., they become equalities when $b_m = e_{rm}/\sqrt{m}$*, where* $r = 2, 1, 3$*, and for* $j \le n$*,*

$$\begin{aligned} e_{1j} &= \sqrt{j}, \\ e_{2j} &= (3j - 2n - 1)\sqrt{j}, \\ e_{3j} &= \big(10j^2 - 6(2n+1)j + 3n^2 + 3n + 2\big)\sqrt{j}. \end{aligned}$$

Vectors e_1, e_2 and e_3 are orthogonal eigenvectors of matrix $W = ((\mathbf{1}\{i + j > n\}(ij)^{-1/2}))$, where $i, j \le n$ and $n \ge 3$, corresponding to the three absolutely largest eigenvalues. The statement of Lemma 2.4 follows from applying these facts to the Rayleigh quotient of matrix W. For full proof of the lemma, see [19].

2.2 Estimation of quadratic forms

We begin by estimating the quadratic form Δ_{n1}.

Proposition 2.5. *Given an arbitrary vector* $\bar{a} \in \mathbb{R}^n$ *with* $n \ge 2$*,*

$$\Delta_{n1}(\bar{a}) \ge -\frac{1}{2} D_n(\bar{a}) \ge -\frac{1}{2} B_n(\bar{a}).$$

If $\bar{a}_2$ *is a vector with coordinates* $a_{2j} := (3j - 2n - 1)p_j(n)^{-1}$*,* $j \le n$*, then*

$$\Delta_{n1}(\bar{a}_2) = -\frac{1}{2} B_n(\bar{a}_2)\Big(1 + O\Big(\frac{1}{n}\Big)\Big). \tag{2.3}$$

If $\bar{a}$ *is a solution to the condition*

$$\sum_{j \le n} j x_j p_j(n) = 0, \tag{2.4}$$

then

$$\Delta_{n1}(\bar{a}) \le \frac{1}{3} B_n(\bar{a}).$$

Furthermore, if $a_j = a_{3j} = e_{3j} j^{-1/2} p_j(n)^{-1}$*,* $j \le n$*, then*

$$\Delta_{n1}(\bar{a}_3) = \frac{1}{3} B_n(\bar{a}_3)\Big(1 + O\Big(\frac{1}{n}\Big)\Big).$$

Proof. Applying Lemma 2.4 for $b_j = a_j p_j(n)$ if $j \le n$, we obtain the first inequality of Proposition 2.5. As stated in Lemma 2.4, the equality is achieved if $\bar{a} = \bar{a}_2$ as given above.

Justification of the relation (2.3) is a straightforward calculation. For $j \le n/2$, we have $p_j(n) = j^{-1}(1 - (2j)^{-1} + O(j^{-2}))$, and hence

$$\frac{1}{jp_j(n)^2} - j = \frac{j}{1 - j^{-1} + O(j^{-2})} - j = 1 + O(j^{-1}).$$

On the other hand, for $j > n/2$, $1/(jp_j(n)^2) - j$ vanishes. Thus it is uniformly $O(1)$ for $j \le n$. Using the expression derived above, we compare $\Delta_{n1}(\bar{a}_2)$ and $\frac{1}{2}B_n(\bar{a}_2)$:

$$\Delta_{n1}(\bar{a}_2) + \frac{1}{2}B_n(\bar{a}_2) = \frac{1}{2}\sum_{j\le n}(3j-2n-1)^2\left(\frac{1}{jp_j(n)^2} - j\right) = O(n^3).$$

Moreover, summing explicitly yields

$$\begin{aligned}\Delta_{n1}(\bar{a}_2) &= \sum_{\substack{i,j\le n\\ i+j>n}} a_{2i}a_{2j}p_i(n)p_j(n) = -\frac{1}{2}\sum_{j\le n}(3j-2n-1)^2 j\\ &= -\frac{n(n-1)(n+1)(n+2)}{8}.\end{aligned}$$

Since $\Delta_{n1}(\bar{a}_2) = -n^4/8 + O(n^3)$ implies $B_n(\bar{a}_2)/2 = n^4/8 + O(n^3)$, we arrive at claim (2.3). To prove the last part of Proposition 2.5, we firstly check that the vector $\bar{a}_3$ satisfies equation (2.4). Indeed, $\bar{e}_1$ is orthogonal to $\bar{e}_3$. In other words,

$$0 = \bar{e}_1\bar{e}_3' = \sum_{j\le n}\sqrt{j}\big(a_{3j}\sqrt{j}p_j(n)\big),$$

which gives the claim. It remains to find the asymptotic behavior of $B_n(\bar{a}_3)$ as $n \to \infty$ and compare it to $\Delta_{n1}(\bar{a}_3)$.

$$-\Delta_{n1}(\bar{a}_3) + \frac{1}{3}B_n(\bar{a}_3) = \frac{1}{3}\sum_{j\le n}(e_{3j})^2 j^{-1}\left(\frac{1}{jp_j(n)^2} - j\right) = O(n^5).$$

The estimate above, together with $\Delta_{n1}(\bar{a}_3) = n^6/18 + O(n^5)$ gives us the last claim. $\square$

Now, the task is to compare the quadratic forms $\Delta_{n2}(\bar{a})$ and $D_n(\bar{a})$. The substitution $a_j = x_j\sqrt{p_j(n)}$, $j \le n$, reduces them to $\bar{x}Q_n\bar{x}'$ and $||\bar{x}||^2$ respectively. Here $\bar{x} = (x_1, \ldots, x_n)$, $\bar{x}'$ is its transpose, and $Q_n = ((q_{ij}))$ is the matrix with entries

$$q_{ij} = \mathbf{1}\{i + j \le n\}\frac{p_i(n)p_j(n) - p_{i,j}(n)}{\sqrt{p_i(n)p_j(n)}}, \quad i, j \le n.$$

If λ_j, $j \le n$, are the eigenvalues of Q_n, then the inequality $\left|\bar{x}Q_n\bar{x}'\right| \le \max_{j\le n}|\lambda_j|\,||\bar{x}||^2$ implements our task.

Proposition 2.6. *We have*

$$\mathrm{Trace}(Q_n^2) = \sum_{j\le n}\lambda_j^2 \ll n^{-2/3}.$$

Consequently,

$$\Delta_{n2}(\bar{a}) \ll n^{-1/3}D_n(\bar{a}), \quad \bar{a}\in\mathbb{R}^n.$$

We will estimate the sum

$$\mathrm{Trace}(Q_n^2) = \sum_{\substack{i+j\le n\\ i\ne j}} \frac{\big(p_i(n)p_j(n) - p_{i,j}(n)\big)^2}{p_i(n)p_j(n)},$$

summing separately over (i,j) belonging to the different regions:

$$\begin{gathered} T_1 := \{(i,j):\ i,j\le n/2,\, i\ne j\},\\ T_2 := \{(i,j):\ n/2 < i \le n,\, n/K < j \le n-i\},\\ T_3 := \{(i,j):\ n/2 < i \le n(1-2/K),\, j\le n/K\},\\ T_4 := \{(i,j):\ n(1-2/K) < i \le n,\, j \le \min\{n-i, n/K\}\}, \end{gathered}$$

and

$$T_5 := \{(i,j):\ i\le n-j,\, n/2 < j \le n,\, i\ne j\}.$$

Here $4\le K<n$ is a parameter to be chosen later. Set

$$\Sigma_m := \sum_{(i,j)\in T_m}\frac{(p_i(n)p_j(n) - p_{i,j}(n))^2}{p_i(n)p_j(n)}, \quad 1\le m\le 5.$$

Let us start with estimations of appropriate sums involving the remainders

$$r_j(n) := \sum_{k>n/j}\frac{(-1)^k}{j^k k!} = p_j(n) - 1 + e^{-1/j}, \quad j\le n,$$

and

$$r_{i,j}(n) := \sum_{\substack{il+jk>n\\ l,k\ge 1}}\frac{(-1)^{l+k}}{i^l l! j^k k!} = p_{i,j}(n) - (1-\mathrm{e}^{-1/i})(1-\mathrm{e}^{-1/j}), \quad i\ne j.$$

The inequalities

$$\mathrm{e}^{1/j}-1\le \mathrm{e}/j, \quad \mathrm{e}^{1/j}-1-1/j\le \mathrm{e}/(2j^2), \quad 1/(2j)\le p_j(n)\le 1/j, \quad j\ge 1$$

will be frequently used without recalling.

Lemma 2.7. *If $j \le n/2$, then*

$$|r_j(n)| \le \frac{\mathrm{e}^3}{\sqrt{6\pi}} \frac{1}{n^3}. \tag{2.5}$$

Furthermore,

$$\sum_{i,j \le n/2} i^{-1} j r_j(n)^2 \le \frac{\mathrm{e}^6 \log(2\mathrm{n})}{24\pi n^4}.$$

Proof. Directly, if $m_j := \lfloor n/j \rfloor + 1$, then

$$|r_j(n)| \le \left| \sum_{k > n/j} \frac{(-1)^k}{j^k k!} \right| \le \frac{1}{j^{m_j} m_j!} \le \frac{1}{\sqrt{2\pi m_j}} \Big(\frac{\mathrm{e}}{j m_j} \Big)^{m_j}.$$

Thus, if $j \le n/2$ and $n \ge 4$, then $m_j \ge 3$ and

$$|r_j(n)| \le \frac{1}{\sqrt{6\pi}} \Big(\frac{\mathrm{e}}{j(n/j)} \Big)^3 \le \frac{\mathrm{e}^3}{\sqrt{6\pi}} \frac{1}{n^3}$$

as desired.

Since $\sum_{i \le x} 1/i \le \log x + 1$, $x > 1$, the second estimate in Lemma 2.7 follows from the first one. $\square$

Lemma 2.8. *If $n \ge 4$, then*

$$\sum_{\substack{i,j \le n/2 \\ i \ne j}} i j r_{i,j}(n)^2 \ll \frac{\log n}{n^2}.$$

Proof. Observe that $i, j \le n/2$, $i \ne j$, and $il + jk > n$ imply $l \ge 2$ or $k \ge 2$. Consequently,

$$\begin{aligned} |r_{i,j}(n)| &\le \frac{\mathbf{1}\{2i + j > n\}}{2i^2 j} + \frac{\mathbf{1}\{i + 2j > n\}}{2ij^2} + \sum_{\substack{il + jk > n \\ l,k \ge 2}} \frac{1}{i^l l! j^k k!} \\ &< \frac{\mathbf{1}\{2i + j > n\}}{2i^2 j} + \frac{\mathbf{1}\{i + 2j > n\}}{2ij^2} + \frac{1}{n} \sum_{l,k \ge 2} \frac{il + jk}{i^l l! j^k k!} \\ &\le \frac{\mathbf{1}\{2i + j > n\}}{2i^2 j} + \frac{\mathbf{1}\{i + 2j > n\}}{2ij^2} + \frac{\mathrm{e}^2}{2n} \frac{i + j}{i^2 j^2}. \end{aligned}$$

Consequently, using $|a + b|^2 \le 2|a|^2 + |b|^2$, $a, b \in \mathbb{R}$, repetitively, we obtain

$$\sum_{\substack{i,j \le n/2 \\ i \ne j}} i j r_{i,j}(n)^2 \ll \sum_{\substack{i,j \le n/2 \\ i + 2j > n}} \frac{1}{ij^3} + \frac{1}{n^2} \sum_{i,j \le n/2} \frac{1}{ij^3}$$

$$\ll \sum_{i\le n/2} \frac{1}{i} \sum_{j>n/4} \frac{1}{j^3} + \frac{\log n}{n^2} \ll \frac{\log n}{n^2}.$$

$\square$

Lemma 2.9. *If $n \ge 4$, then*

$$\Sigma_1 \ll \frac{\log n}{n^2}.$$

Proof. By the definitions,

$$\begin{aligned}\big|p_i(n)p_j(n) - p_{i,j}(n)\big| &= \big|(1-\mathrm{e}^{-1/i})r_j(n) + (1-\mathrm{e}^{-1/j})r_i(n)\\ &\qquad + r_i(n)r_j(n) - r_{i,j}(n)\big|\\ &\le i^{-1}|r_j(n)| + j^{-1}|r_i(n)|\\ &\qquad + |r_i(n)r_j(n)| + |r_{i,j}(n)|.\end{aligned}$$

Applying Lemmata 2.7 and 2.8, we obtain

$$\begin{aligned}\Sigma_1 &\ll \sum_{i,j\le n/2} i^{-1}jr_j(n)^2 + \Big(\sum_{i\le n/2} ir_i(n)^2\Big)^2 + \sum_{i,j\le n/2} ijr_{i,j}(n)^2\\ &\ll \frac{\log n}{n^4} + \frac{\log n}{n^2} \ll \frac{\log n}{n^2}.\end{aligned}$$

$\square$

Lemma 2.10. *Let $4 \le K \le n$ be arbitrary, then*

$$\Sigma_2 \ll \frac{K^2}{n^2}.$$

Proof. If $n/2 < i \le n$ and $i + j \le n$, then

$$p_i(n) = \frac{1}{i}, \quad p_{i,j}(n) = \frac{1}{i}p_j(n-i), \tag{2.6}$$

and

$$p_j(n) - p_j(n-i) = \sum_{n-i<jk\le n} \frac{(-1)^{k+1}}{j^k k!} \ll \frac{1}{j^2}. \tag{2.7}$$

We had used the fact that the summand corresponding to $k = 1$ was missing for the above indicated i and j. Therefore,

$$\Sigma_2 = \sum_{(i,j)\in T_2} \frac{1}{i}\frac{(p_j(n)-p_j(n-i))^2}{p_j(n)} \ll \sum_{n/2<i\le n} \frac{1}{i} \cdot \sum_{j>n/K} \frac{1}{j^3} \ll \frac{K^2}{n^2}.$$

$\square$

Lemma 2.11. *If $n \geq 4$, then*

$$\Sigma_3 = \sum_{(i,j)\in T_3} \frac{1}{i} \frac{(p_j(n) - p_j(n-i))^2}{p_j(n)} \ll \frac{K^4 \log n}{n^4}.$$

Proof. The equality follows from (2.6). If $j \leq n/K$ and $i \leq n(1-2/K)$, then $j \leq (n-i)/2$. Hence, by (2.5) in Lemma 2.7 with $n-i$ instead of n, we have

$$(p_j(n) - p_j(n-i))^2 \ll r_j(n)^2 + r_j(n-i)^2 \ll \frac{1}{n^6} + \frac{1}{(n-i)^6} \ll \frac{K^6}{n^6}.$$

Therefore,

$$\Sigma_3 \ll \sum_{(i,j)\in T_3} \frac{j}{i}(p_j(n) - p_j(n-i))^2 \ll \frac{n^2}{K^2}\frac{K^6}{n^6} \log n = \frac{K^4 \log n}{n^4}.$$

□

Lemma 2.12. *If $4 \leq K \leq n/2$, then*

$$\Sigma_4 \ll \frac{1}{K}.$$

Proof. Again applying (2.6) and (2.7), we obtain

$$\Sigma_4 = \sum_{(i,j)\in T_4} \frac{j}{i}(p_j(n) - p_j(n-i))^2 \ll \sum_{\substack{n(1-2/K)<i\leq n \\ j\leq \min\{n-i,n/K\}}} \frac{1}{ij^3}$$

$$\ll \sum_{n(1-2/K)<i\leq n} \frac{1}{i} \ll \log \frac{n}{n(1-2/K)} + \frac{1}{n(1-2/K)} \ll \frac{1}{K},$$

if $4 \leq K \leq n/2$. □

Proof of Proposition 2.6. By symmetry

$$\Sigma_5 = \Sigma_2 + \Sigma_3 + \Sigma_4.$$

Adding all estimates of Σ_m, $m \leq 5$, we have

$$\mathrm{Trace}(Q_n^2) \ll \frac{\log n}{n^2} + \frac{K^2}{n^2} + \frac{K^4 \log n}{n^4} + \frac{1}{K}.$$

provided that $4 \leq K \leq n/2$. Now the choice $K = n^{2/3}$, for n sufficiently large, gives

$$\mathrm{Trace}(Q_n^2) \ll n^{-2/3}.$$

□

Proof of Theorem 1.1. We apply Propositions 2.5 and 2.6 to Δ_{n1} and Δ_{n2} in the variance expression (1.4). $\square$

Chapter 3

Variance bound refinement

In this chapter, we prove Theorem 1.2 stated in Section 1.9.1, using the previously introduced notation.

3.1 Auxiliary lemmata

Lemma 3.1. *Let M be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. For every real a and vector $\bar{x} \in \mathbb{R}^n$, we have*

$$\min_{i \leq n} |\lambda_i - a| \cdot ||\bar{x}|| \leq ||\bar{x}M - a\bar{x}||,$$

where $||\bar{z}||$ denotes the Euclidean norm of a vector $\bar{z} \in \mathbb{R}^n$.

For proof, see Lemma 20.5 in [8].

The value sums of appearing functions will be approximated by the Euler-Maclaurin formula.

Lemma 3.2. *If $a, b \in \mathbb{N}_0$ and $f \in \mathcal{C}^2[a, b]$, then*

$$\sum_{a<j\leq b} f(j) - \int_a^b f(u)\mathrm{d}u \ll \big|f(b) - f(a)\big| + \big|f'(b) - f'(a)\big| + \int_a^b \big|f''(u)\big|\mathrm{d}u.$$

For proof, see Theorem 0.7 on page 7 of the book [45].

We owe to J. Kubilius [25], who has proposed to use the Jacobi polynomials $P_{r-1}^{(1,0)}(u)$, $r \in \mathbb{N}$, to construct approximations for the eigenvectors of specific matrices. Namely, we will apply the sequence of polynomials

$$\begin{aligned}\psi_r(u) &:= uP_{r-1}^{(1,0)}(1-2u)\\ &= u\sum_{0\le k\le r-1}(-1)^{r-1-k}\binom{r}{k}\binom{r-1}{k}u^{r-1-k}(1-u)^k\end{aligned}$$

starting with $\psi_1(u)=u$, $\psi_2(u)=u(2-3u)$ etc.The required properties of the aforementioned polynomials are listed below.

Lemma 3.3. *For every $r,s\in\mathbb{N}$ and $0\le u\le 1$, we have*

$$\int_{1-u}^{1}\psi_r(v)v^{-1}\mathrm{d}v=\lambda_r\psi_r(u),\quad \lambda_r=(-1)^{r-1}r^{-1},$$

$$\int_0^1\psi_r(v)\psi_s(v)v^{-1}\mathrm{d}v=\begin{cases}0, & \text{if } r\ne s\\ (2r)^{-1}, & \text{if } r=s\end{cases}$$

and

$$\psi_r(u)\ll u,\quad |\psi_r'(u)|+|\psi_r''(u)|\ll 1,\quad \psi_r(2u)-2\psi_r(u)\ll u^2.$$

For proof see [25]. We just add that the last estimate follows from the formula

$$\psi_r(2u)-2\psi_r(u)=2\lambda_r u\int_{1-u}^{1-2u}\psi_r''(v)\mathrm{d}v,$$

which stems from the first relation of the lemma.

Actually, we will use the following discrete version of the last lemma.

Lemma 3.4. *Let $\psi_r(u)$, $0\le u\le 1$, and λ_r, $1\le r\le n$, be as in Lemma 3.3. Then*

$$\sum_{n-j<i\le n}\frac{\psi_r(i/n)}{i}=\lambda_r\psi_r(j/n)+O\Big(\frac{j}{n^2}\Big),\quad j\le n. \tag{3.1}$$

Furthermore,

$$\sum_{j\le n}\frac{\psi_r^2(j/n)}{j}=\frac{1}{2r}+O\Big(\frac{1}{n}\Big).$$

Proof. Let $j\le n-1$. Then, by Lemma 3.2,

$$\sum_{n-j<i\le n}\frac{\psi_r(i/n)}{i}-\int_{n-j}^{n}\frac{\psi_r(t/n)}{t}\mathrm{d}t\ll\Big|\frac{\psi_r(1)}{n}-\frac{\psi_r(1-j/n)}{n-j}\Big|$$

$$+\left|\frac{\psi_r'(1)-\psi_r(1)}{n^2}\right|+\left|\frac{\psi_r'(1-j/n)}{n(n-j)}-\frac{\psi_r(1-j/n)}{(n-j)^2}\right|$$
$$+\int_{n-j}^{n}\left|\left(\frac{\psi_r(u/n)}{u}\right)''\right|\mathrm{d}u \ll \frac{1}{n}. \tag{3.2}$$

If $j = n$, separating the summand $\psi_r(1)/n \ll 1/n$ and repeating the just exposed evaluations for the remaining sum, we would have relation (3.2) again.

If $j \le n/2$, due to

$$\psi_r(1-j/n)/(1-j/n) = \psi_r(1)\big(1+O(j/n)\big)$$

and

$$\int_{n-j}^{n}\left|\left(\frac{\psi_r''(u/n)}{n^2u}\right)-2\frac{\psi_r'(u/n)}{nu^2}+\frac{2\psi_r(u/n)}{u^3}\right|\mathrm{d}u \ll \frac{j}{n^2},$$

we obtain the estimate of errors as claimed in (3.1). Since by Lemma 3.3 the integral on the left-hand side of (3.2) equals $\lambda_r\psi_r(j/n)$. □

The vectors $\bar{a}_r$ with coordinates $a_{rj} := \psi_r(j/n)$, where $j, r \le n$, will play the essential role in the proof of Theorem 1.2. As a corollary from Lemma 3.4, we state the first simple observation.

Lemma 3.5. *We have*

$$D_n(\bar{a}_r) = B_n(\bar{a}_r) + O(n^{-1}) = (2r)^{-1} + O(n^{-1}), \qquad \Delta_{n3}(\bar{a}_r) \ll n^{-1}.$$

Proof. Apply Lemmata 3.3 and 3.4 to get

$$D_n(\bar{a}_r) = \sum_{j\le n}\psi_r^2(j/n)\left(\frac{1}{j}+O\Big(\frac{1}{j^2}\Big)\right) = (2r)^{-1} + O(n^{-1}).$$

The same argument suffices for the other quadratic forms. □

3.2 The first iteration

We now explore the quadratic forms $\Delta_{n1}(\bar{a})$, $\Delta_{n2}(\bar{a})$ and their sum. Denote the transform

$$a_j = x_j\sqrt{j}, \quad j \le n,$$

where $\bar{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$ is arbitrary and $\bar{x}'$ is its transpose. Then the quadratic form

$$\Delta_{n1}(\bar{a}) + \Delta_{n2}(\bar{a})$$

reduces to the congruent one, namely, to

$$\bar{x}Q\bar{x}' = \bar{x}\widehat{Q}\bar{x}' + \bar{x}\widetilde{Q}\bar{x}',$$

where Q is the symmetric matrix with entries

$$\begin{aligned} q_{ij} :=& \hat{q}_{ij} + \tilde{q}_{ij} := \mathbf{1}\{i, j \le n, i + j > n\} p_i(n) p_j(n) \sqrt{ij} \\ &+ \mathbf{1}\{i + j \le n,\ i \ne j\} \big(p_i(n) p_j(n) - p_{ij}(n)\big) \sqrt{ij}, \quad i, j \le n. \end{aligned}$$

The goal of this section is to find the first iteration for the eigenvalues $\mu_r := \mu_r(n)$, $r \le n$, of the matrix Q. The outcome is presented in Proposition 3.7 below. So far, we know that the minimal eigenvalue of Q is in the interval of length $O(n^{-1/3})$ around the point $-1/2$. Now this bound will be sharpened up to $O(n^{-1} \log n)$. In the next section, we will construct the second iteration, which further improves this result. Somewhat intricate calculations are presented in a few lemmata.

The following transforms of a vector $\bar{x} \in \mathbb{R}^n$ are used afterward:

$$\begin{aligned} b_{1j}(\bar{x}) &:= \frac{\mathbf{1}\{j \le n/2\} x_j}{j}, \\ b_{2j}(\bar{x}) &:= \frac{1}{\sqrt{j}} \sum_{n-j<i\le n/2} \frac{x_i}{i\sqrt{i}}. \end{aligned}$$

where $j \le n$.

Simply, let $\bar{\rho}(n)$ be a vector, depending at most on r, with coordinates $\rho_j(n)$, $j \le n$, not necessarily identical in different instances, and satisfying the condition

$$||\bar{\rho}(n)||^2 = \sum_{j\le n} \rho_j(n)^2 = O(n^{-2}).$$

Moreover, set $(\bar{x}M)_j$, $j \le n$, for the jth coordinate of the transformed vector $\bar{x}M$, where M is a $n \times n$ matrix.

Lemma 3.6. *Define $\bar{y}_r = (y_{r1}, \ldots, y_{rn})$ and $y_{rj} := \psi_r(j/n)/\sqrt{j}$. Then for $j \le n$,*

$$(\bar{y}_r\widehat{Q})_j = \lambda_r y_{rj} - \frac{\lambda_r}{2} b_{1j}(\bar{y}_r) - \frac{1}{2} b_{2j}(\bar{y}_r) + \rho_j(n)$$

$$= \lambda_r y_{rj} + O\Big(\frac{1 + \mathbf{1}\{n/2 < j \le n\}(\log n)}{n\sqrt{j}}\Big) + \rho_j(n) \qquad (3.3)$$

and

$$(\bar{y}_r\widetilde{Q})_j = -\frac{\lambda_r}{2} b_{1j}(\bar{y}_r) + \rho_j(n) = O\Big(\frac{1}{n\sqrt{j}}\Big) + \rho_j(n). \qquad (3.4)$$

Proof. By the definitions and Lemma 3.4,

$$\begin{aligned}
(\bar{y}_r\widehat{Q})_j &:= \sum_{i \le n} y_{ri}\hat{q}_{ij} = p_j(n)\sqrt{j} \sum_{n-j<i\le n} \psi_r(i/n)p_i(n) \\
&= \Big(\frac{1}{j} - \frac{\mathbf{1}\{j \le n/2\}}{2j^2} + O\Big(\frac{1}{j^3}\Big)\Big)\sqrt{j} \sum_{n-j<i\le n} \psi_r(i/n)p_i(n) \\
&= \Big(\frac{1}{\sqrt{j}} - \frac{\mathbf{1}\{j \le n/2\}}{2j\sqrt{j}}\Big) \\
&\quad \cdot \sum_{1\le s<n/(n-j)} \frac{(-1)^{s+1}}{s!} \sum_{n-j<i\le n/s} \frac{1}{i^s}\psi_r(i/n) + \rho_j(n) \\
&= \Big(\frac{1}{\sqrt{j}} - \frac{\mathbf{1}\{j \le n/2\}}{2j\sqrt{j}}\Big) \\
&\quad \cdot \Big[\sum_{n-j<i\le n} \frac{\psi_r(i/n)}{i} - \frac{1}{2}\sum_{n-j<i\le n/2} \frac{1}{i^2}\psi_r(i/n)\Big] + \rho_j(n). \qquad (3.5)
\end{aligned}$$

In the last step, we have used the fact that the sum over $3 \le s \le n/(n-j)$ concerned only $j \in [2n/3, n]$.

If $j \le n/2$, then $b_{2j}(\bar{y}_r) = 0$, thus (3.5) can be rewritten as

$$\begin{aligned}
(\bar{y}_r\widehat{Q})_j &= \Big(\frac{1}{\sqrt{j}} - \frac{1}{2j\sqrt{j}}\Big)\big(\lambda_r\psi_r(j/n) + \rho_j(n)\big) \\
&= \lambda_r y_{rj} - \frac{\lambda_r}{2} b_{1j}(\bar{y}_r) + \rho_j(n) \\
&= \lambda_r y_{rj} + O\Big(\frac{1}{n\sqrt{j}}\Big) + \rho_j(n).
\end{aligned}$$

This coincides with the expected formula (3.3) in this instance.

If $n/2 < j \le n$, then (3.5) implies

$$\begin{aligned}
(\bar{y}_r\widehat{Q})_j &= \lambda_r y_{rj} + O\Big(\frac{\mathbf{1}\{n/2 < j \le n\}}{n\sqrt{j}}\Big) \\
&\quad - \frac{1}{2\sqrt{j}} \sum_{n-j<i\le n/2} \frac{1}{i^2}\psi_r(i/n) + \rho_j(n)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_r y_{rj} - \frac{1}{2} b_{2j}(\bar{y}_r) + \rho_j(n) \\
&= \lambda_r y_{rj} + O\Big(\frac{\mathbf{1}\{n/2 < j \le n\}(\log n)}{n\sqrt{j}}\Big) + \rho_j(n)
\end{aligned}
$$

as desired. Relation (3.3) is established.

Now we examine $\bar{y}_r \widetilde{Q}$. The jth coordinate equals

$$
\begin{aligned}
\big(\bar{y}_r \widetilde{Q}\big)_j &:= \sqrt{j} \sum_{i \le n-j,\, i \ne j} \psi_r(i/n)[p_i(n) p_j(n) - p_{i,j}(n)] \\
&= \sqrt{j} \sum_{i \le n-j,\, i \ne j} \psi_r(i/n) \Bigg(\frac{1}{i} \sum_{(n-i)/j < k \le n/j} \frac{(-1)^{k+1}}{j^k k!} \\
&\quad + \sum_{\substack{2 \le l \le n/i \\ (n-il)/j < k \le n/j}} \frac{(-1)^{l+k}}{i^l l! j^k k!} \Bigg) \\
&=: S_{1j} + S_{2j}.
\end{aligned}
$$

Here $S_{1j} = 0$ if $j > n/2$, otherwise,

$$
\begin{aligned}
S_{1j} &= \mathbf{1}\{j \le n/2\} \sqrt{j} \sum_{2 \le k \le n/j} \frac{(-1)^{k+1}}{j^k k!} \sum_{\substack{n-jk < i \le n-j, \\ i \ne j}} \frac{\psi_r(i/n)}{i} \\
&= -\frac{\mathbf{1}\{j \le n/2\}}{2j\sqrt{j}} \sum_{\substack{n-2j < i \le n-j, \\ i \ne j}} \frac{\psi_r(i/n)}{i} \\
&\quad + O\Bigg(\frac{\sqrt{j}}{n} \sum_{3 \le k \le n/j} \frac{1}{j^{k-1}(k-1)!} \Bigg) \\
&= -\frac{\mathbf{1}\{j \le n/2\}}{2j\sqrt{j}} \sum_{n-2j < i \le n-j} \frac{\psi_r(i/n)}{i} + \rho_j(n).
\end{aligned}
$$

By Lemmata 3.3 and 3.4,

$$
\begin{aligned}
\sum_{n-2j < i \le n-j} \frac{\psi_r(i/n)}{i} &= \lambda_r \big(\psi_r(2j/n) - \psi_r(j/n)\big) + O(j/n^2) \\
&= \lambda_r \psi_r(j/n) + O(j^2/n^2).
\end{aligned}
$$

Hence,

$$
S_{1j} = -\frac{\lambda_r \mathbf{1}\{j \le n/2\}}{2j\sqrt{j}} \psi_r(j/n) + \rho_j(n) = -\frac{\lambda_r}{2} b_{1j}(\bar{y}_r) + \rho_j(n). \tag{3.6}
$$

Let us examine the next sum

$$S_{2j} = \sqrt{j} \sum_{i \le n-j,\, i \ne j} \psi_r(i/n) \Bigg(\sum_{\substack{2 \le l \le n/i, \\ (n-il)/j < k \le n/j}} \frac{(-1)^{l+k}}{i^l l! j^k k!} \Bigg).$$

The summand corresponding to the pair $(l,k) = (2,1)$ in the inner sum contributes to S_{2j} the quantity not exceeding

$$\begin{aligned} &\frac{\mathbf{1}\{j \le n/2\}}{2\sqrt{j}} \sum_{(n-j)/2 < i \le n/2} \frac{1}{i^2} \big|\psi_r(i/n)\big| \\ &\quad + \frac{\mathbf{1}\{n/2 < j \le n\}}{2\sqrt{j}} \sum_{(n-j)/2 < i \le n-j} \frac{1}{i^2} \big|\psi_r(i/n)\big| \\ &\ll \frac{\mathbf{1}\{j \le n/2\}}{n\sqrt{j}} \log(1 - j/n)^{-1} + \frac{\mathbf{1}\{n/2 < j \le n\}}{n\sqrt{j}} \\ &\ll \sqrt{j} n^{-2} + n^{-3/2} = \rho_j(n). \end{aligned} \tag{3.7}$$

For the remaining summands in S_{2j}, we have $il, jk \le n$ with $l, k \ge 2$. The additional summand corresponding to $i = j$ would contribute the $\rho_j(n)$ error. Splitting the completed sum into two parts, we get

$$\begin{aligned} S_{2j} &\ll \rho_j(n) + \frac{\mathbf{1}\{j \le n/2\}\sqrt{j}}{n} \sum_{\substack{2 \le k \le n/(2j) \\ 2 \le l \le n}} \frac{1}{j^k k! l!} \sum_{(n-jk)/l < i \le n/l} \frac{1}{i^{l-1}} \\ &\quad + \frac{\mathbf{1}\{j \le n/2\}\sqrt{j}}{n} \sum_{\substack{n/(2j) < k \le n/j \\ 2 \le i \le n}} \frac{1}{j^k k! l!} \sum_{(n-jk)/l < i \le n-j} \frac{1}{i^{l-1}} \\ &\ll \rho_j(n) + \frac{\mathbf{1}\{j \le n/2\}\sqrt{j}}{n} \sum_{2 \le k \le n/(2j)} \frac{1}{j^k k!} \log\left(1 - \frac{jk}{n}\right)^{-1} \\ &\quad + \frac{\mathbf{1}\{j \le n/2\}\sqrt{j}\log n}{n} \sum_{n/(2j) < k \le n/j} \frac{1}{j^k k!}. \end{aligned}$$

The logarithmic inequality suffices to estimate the first sum on the right-hand side. The last inner sum

$$\begin{aligned} &\Big(\mathbf{1}\{j \le n/\log n\} + \mathbf{1}\{n/\log n < j \le n/2\}\Big) \sum_{n/(2j) < k \le n/j} \frac{1}{j^k k!} \\ &\ll \sum_{k > (\log n)/2} \frac{1}{k!} + \frac{\log^2 n}{n^2} \ll \frac{\log^2 n}{n^2}. \end{aligned} \tag{3.8}$$

This and (3.7) yield the estimate $S_{2j} = \rho_j(n)$ for $j \le n$. Consequently, taking into account (3.6), we obtain

$$\left(\bar{y}_r \widetilde{Q}\right)_j = -\frac{\lambda_r}{2} b_{1j}(\bar{y}_r) + \rho_j(n), \quad j \le n.$$

This is the first relation in (3.4) leading to the second rougher estimate. $\square$

Proposition 3.7. *Let* $\mathrm{Sp}(Q)$ *be the spectrum of the matrix* Q *and, as above,* $\lambda_r = (-1)^{r-1}/r$. *For each fixed* $r \ge 1$, *there exist constants* C_r *and* $n_r \in \mathbb{N}$, *and a unique eigenvalue* $\mu_r \in \mathrm{Sp}(Q)$ *such that*

$$|\mu_r - \lambda_r| \le C_r (\log n)/n \tag{3.9}$$

provided that $n \ge n_r$.

Proof. To prove the existence of an eigenvalue in the vicinity of λ_r, it suffices to apply Lemma 3.1. Namely, one has to sum up over $j \le n$ the squares of errors on the right-hand sides of (3.3) and (3.4) to get $O\big((\log n)/n\big)$ bound. Since by Lemma 3.4 we have $||\bar{y}_r||^2 = 1/(2r) + O(n^{-1})$, the vectors $\bar{y}_r$ become the approximate eigenvectors for each μ_r lying close to λ_r.

Let us prove the claimed uniqueness. Suppose that μ_m, a neighbor of λ_m, $m \le n$, is the first having a partner $\mu_{m'}$ such that $\mu_{m'} - \mu_m \ll_{m,m'} (\log n)/n$ with $m' > m$ and let n be large. Fix another index $r_0 > 4m^2$ and introduce

$$C := \max\{C_r : \ r \le r_0\}, \qquad n_0 := \max\{n_r : \ r \le r_0\}.$$

Then (3.9) holds for all $r \le r_0$ if C is substituted for C_r and $n \ge n_0$. Furthermore, we can find an $n' > n_0$ such that the intervals $I_r := \big[\lambda_r - C(\log n)/n, \lambda_r + C(\log n)/n\big]$, $r \le r_0$, do not intersect if $n \ge n'$. This implies that $m' > r_0 > 4m^2$. After showing that $|\mu_r| \le 1/r_0 < 1/(2m)$ for each $r > r_0$, when n is sufficiently large, we will be sure that no such $\mu_{m'}$ exists at all.

For this purpose, we evaluate the trace of matrix Q^2. Since $\hat{q}_{ij}\tilde{q}_{ij} = 0$, we have

$$\mathrm{Trace}(Q^2) = \mathrm{Trace}(\widehat{Q}^2) + \mathrm{Trace}(\widetilde{Q}^2).$$

Using $p_j(n) \le 1/j$ for $j \le n$, we obtain

$$\mathrm{Trace}(\widehat{Q}^2) = \sum_{\substack{i+j>n \\ i,j\le n}} ij(p_i(n)p_j(n))^2 \le \sum_{\substack{i+j>n \\ i,j\le n}} \frac{1}{ij} = \sum_{r\le n} \frac{1}{r^2} = \sum_{r\le n} \lambda_r^2.$$

Further, we observe that

$$\begin{aligned}\mathrm{Trace}(\widetilde{Q}^2) &= \sum_{\substack{i+j\le n \\ i\ne j}} ij(p_i(n)p_j(n) - p_{i,j}(n))^2 \\ &\le \sum_{\substack{i+j\le n \\ i\ne j}} \frac{(p_i(n)p_j(n) - p_{i,j}(n))^2}{p_i(n)p_j(n)} = o(1).\end{aligned}$$

In the last step, we applied a vague form of the estimate obtained in Theorem 1.1. Consequently,

$$\mathrm{Trace}(Q^2) = \sum_{r\le n} \mu_r^2 \le \sum_{r\le n} \lambda_r^2 + o(1).$$

Combining this and (3.9), we obtain

$$\begin{aligned}\sum_{r_0<r\le n} \mu_r^2 &\le \sum_{r_0<r\le n} \lambda_r^2 + \sum_{r\le r_0} (\lambda_r^2 - \mu_r^2) + o(1) \\ &= \sum_{r_0<r\le n} \frac{1}{r^2} + o(1) < \int_{r_0}^{\infty} \frac{\mathrm{d}u}{u^2} + o(1) = \frac{1}{r_0} + o(1).\end{aligned}$$

Hence, $\max\{|\mu_r| : \ r > r_0\} < 1/(2m)$. Consequently, apart from μ_m, there is no $\mu_{m'}$ in the interval $I(m)$ if n is sufficiently large. $\square$

3.3 The second iteration

This section further sharpens Proposition 3.7 by dropping the logarithmic factor.

Proposition 3.8. *Recall that $\lambda_r = (-1)^{r-1}/r$ and let μ_r, $r \le n$, be the eigenvalues of Q examined in Proposition* 3.7*. The following approximation is true*

$$\mu_r - \lambda_r \ll n^{-1}$$

for each fixed r.

We now also apply Lemma 3.1 but choose more sophisticated approximate eigenvectors for μ_r. The idea, already adopted in J. Kubilius's number-theoretical paper [26], had been elaborated in effective matrix eigenvalue localization algorithms. Their vast exposition can be found in Chapter 4, [44]. We present the technical calculations in the next lemmata.

Using the previous notation, we introduce the vectors $\bar{b}_1(\bar{y}_r)$ and $\bar{b}_2(\bar{y}_r)$ having the coordinates

$$\begin{aligned} b_{1j}(\bar{y}_r) &= \frac{\mathbf{1}\{j \le n/2\}\psi_r(j/n)}{j\sqrt{j}}, \\ b_{2j}(\bar{y}_r) &= \frac{1}{\sqrt{j}} \sum_{n-j<i\le n/2} \frac{\psi_r(i/n)}{i^2}, \quad j \le n, \end{aligned}$$

respectfully.

Lemma 3.9. *We have $\bar{b}_1(\bar{y}_r)Q = \bar{b}_2(\bar{y}_r) + \bar{\rho}(n)$ for each fixed $r \le n$.*

Proof. By the definitions and recalling that $p_j(n) = 1/j$ if $n/2 < j \le n$, we have

$$\begin{aligned} \big(\bar{b}_1(\bar{y}_r)\widehat{Q}\big)_j &= p_j(n)\sqrt{j} \sum_{n-j<i\le n} b_{1i}(\bar{y}_r)p_i(n)\sqrt{i} \\ &= \frac{\mathbf{1}\{n/2 < j \le n\}}{\sqrt{j}} \sum_{n-j<i\le n/2} \frac{\psi_r(i/n)}{i^2}\Big(1 + O\Big(\frac{1}{i}\Big)\Big) \\ &= b_{2j}(\bar{y}_r) + O\Big(\frac{\mathbf{1}\{n/2 < j \le n\}}{n\sqrt{j}}\Big) = b_{2j}(\bar{y}_r) + \rho_j(n), \quad (3.10) \end{aligned}$$

if $j \le n$.

In the next step, we also follow the proof of Lemma 3.6. Namely,

$$\begin{aligned} \big(\bar{b}_1(\bar{y}_r)\widetilde{Q}\big)_j = \sqrt{j} \sum_{i\le n-j,\, i\ne j} \frac{\mathbf{1}\{i \le n/2\}\psi_r(i/n)}{i} \Big(\frac{1}{i} \sum_{(n-i)/j<k\le n/j} \frac{(-1)^{k+1}}{j^k k!} \\ + \sum_{\substack{2\le l\le n/i,\\ (n-il)/j<k\le n/j}} \frac{(-1)^{l+k}}{i^l l! j^k k!}\Big) = \widetilde{S}_{1j} + \widetilde{S}_{2j}. \end{aligned}$$

Here

$$\widetilde{S}_{1j} = \mathbf{1}\{j \le n/2\}\sqrt{j} \sum_{i\le n/2,\, i\ne j} \frac{\psi_r(i/n)}{i^2} \sum_{(n-i)/j<k\le n/j} \frac{(-1)^{k+1}}{j^k k!}$$

$$= \mathbf{1}\{j \le n/2\}\sqrt{j} \sum_{\substack{2 \le k \le n/j \\ k > n/(2j)}} \frac{(-1)^{k+1}}{j^k k!} \sum_{n-jk<i\le n/2,\, i\ne j} \frac{\psi_r(i/n)}{i^2}$$

$$\ll \frac{\mathbf{1}\{n/4 < j \le n/2\}\log n}{nj^{3/2}} + \frac{\mathbf{1}\{j \le n/4\}\sqrt{j}\log n}{n} \sum_{n/(2j)<k\le n/j} \frac{1}{j^k k!}$$

$$\ll \rho_j(n) + \frac{\sqrt{j}\log^3 n}{n^3} = \rho_j(n), \quad j \le n,$$

by applying (3.8).

The proof of $\widetilde{S}_{2j} = \rho_j(n)$, $j \le n$, is even easier than that in Lemma 3.6. We omit the details. This and (3.10) complete the proof. □

Lemma 3.10. *We have $\bar{b}_2(\bar{y}_r)Q = \bar{\rho}(n)$ for each fixed r.*

Proof. Notice that $b_{2i}(\bar{y}_r) = 0$ if $i \le n/2$, and $p_j(n) = 1/j$ if $n/2 < j \le n$. Using this, we have

$$\begin{aligned} \big(\bar{b}_2(\bar{y}_r)\widehat{Q}\big)_j &= p_j(n)\sqrt{j} \sum_{n-j<i\le n} \mathbf{1}\{i > n/2\} p_i(n) \sum_{n-i<l\le n/2} \frac{\psi_r(l/n)}{l^2} \\ &= p_j(n)\sqrt{j} \sum_{1\le l\le n/2} \frac{\psi_r(l/n)}{l^2} \sum_{(n-j)\vee(n-l)<i\le n} \frac{1}{i}. \end{aligned} \tag{3.11}$$

We divide the sum into parts according to the values of j and l and apply the logarithmic inequality $\log(1-x)^{-1} \le x/2$ if $0 \le x \le 1/2$ where possible. We first obtain

$$\begin{aligned} v_j(n) &:= \mathbf{1}\{n/2 < j \le n\}\big(\bar{b}_2(\bar{y}_r)\widehat{Q}\big)_j \\ &= -\frac{\mathbf{1}\{n/2 < j \le n\}}{\sqrt{j}} \sum_{1\le l\le n/2} \frac{\psi_r(l/n)}{l^2} \sum_{n-l<i\le n} \frac{1}{i} \\ &\ll \frac{1}{\sqrt{n}} \sum_{1\le l\le n/2} \frac{\psi_r(l/n)}{l^2} \log{(1-l/n)^{-1}} \ll n^{-3/2}, \quad j \le n. \end{aligned}$$

Next, we observe that contribution of the partial sum in (3.11) over $l \le j$ is negligible if $j \le n/2$. Indeed,

$$\begin{aligned} w_j(n) &:= \frac{p_j(n)\sqrt{j}\mathbf{1}\{j \le n/2\}}{2} \sum_{1\le l\le j} \frac{\psi_r(l/n)}{l^2} \sum_{n-l<i\le n} \frac{1}{i} \\ &\ll \frac{1}{n\sqrt{j}} \sum_{1\le l\le j} \frac{\psi_r(l/n)}{l} \ll \frac{\sqrt{j}}{n^2}, \quad j \le n. \end{aligned}$$

The last two estimates yield

$$\sum_{j\le n}\big(v_j(n)^2+w_j(n)^2\big)\ll n^{-2}. \tag{3.12}$$

It remains to prove that the vector, which jth coordinate is just the remaining sum in (3.11), namely,

$$\begin{aligned} u_j(n) &:= p_j(n)\sqrt{j}\mathbf{1}\{j\le n/2\}\sum_{j<l\le n/2}\frac{\psi_r(l/n)}{l^2}\sum_{n-j<i\le n}\frac{1}{i}\\ &\ll \frac{\sqrt{j}\mathbf{1}\{j\le n/2\}}{n}\sum_{j<l\le n/2}\frac{\psi_r(l/n)}{l^2},\quad j\le n, \end{aligned}$$

has also small norm. Evaluating the latter, we will apply the Cauchy-Schwarz inequality. So we obtain

$$\begin{aligned} \sum_{j\le n}u_j(n)^2 &= \frac{1}{n^2}\sum_{j\le n/2} j\Big(\sum_{j<l\le n/2}\frac{\psi_r(l/n)}{l^2}\Big)^2\\ &\le \frac{1}{n^2}\sum_{j\le n/2} j(n/2-j)\sum_{j<l\le n/2}\frac{\psi_r^2(l/n)}{l^4}\\ &\ll \frac{1}{n^4}\sum_{j\le n/2} j(n/2-j)\sum_{j<l\le n/2}\frac{1}{l^2}\ll \frac{1}{n^4}\sum_{j\le n/2}(n/2-j)\ll\frac{1}{n^2}. \end{aligned}$$

Combining this, (3.12) and (3.11), we obtain the estimate

$$\bar b_2(\bar y_r)\widehat{Q}=\bar\rho(n). \tag{3.13}$$

While estimating $\big(\bar b_2(\bar y_r)\widetilde{Q}\big)_j$, we first change the order of summation and arrive at

$$\begin{aligned} \big(\bar b_2(\bar y_r)\widetilde{Q}\big)_j &= \sqrt{j}\sum_{i\le n-j,\, i\ne j}\mathbf{1}\{i>n/2\}[p_i(n)p_j(n)-p_{i,j}(n)]\\ &\quad\cdot\sum_{n-i<l\le n/2}\frac{\psi_r(l/n)}{l^2}\\ &= \sqrt{j}\sum_{j<l\le n/2}\frac{\psi_r(l/n)}{l^2}\sum_{n-l<i\le n-j}[p_i(n)p_j(n)-p_{i,j}(n)]. \end{aligned}$$

Here we have left only the nonzero summands and omitted the empty condition $i\ne j$. Afterward, we use the following property:

$$p_i(n)p_j(n)-p_{i,j}(n)=\frac{p_j(n)-p_j(n-i)}{i}=\frac{1}{i}\sum_{(n-i)/j<k\le n}\frac{(-1)^{k+1}}{j^k k!},$$

if $n/2 < i \le n$ and $j \le n$. Thus,

$$
\begin{aligned}
\big(\bar{b}_2(\bar{y}_r)\widetilde{Q}\big)_j &= \sqrt{j} \sum_{j<l\le n/2} \frac{\psi_r(l/n)}{l^2} \sum_{n-l<i\le n-j} \frac{1}{i} \sum_{(n-i)/j<k\le n} \frac{(-1)^{k+1}}{j^k k!} \\
&\ll \frac{1}{j\sqrt{j}} \sum_{j<l\le n/2} \frac{\psi_r(l/n)}{l^2} \sum_{n-l<i\le n-j} \frac{1}{i} \\
&\ll \frac{1}{j\sqrt{j}} \sum_{j<l\le n/2} \frac{\psi_r(l/n)}{l^2} \frac{l-j}{n-j} \\
&\ll \frac{1}{nj\sqrt{j}} \sum_{j<l\le n/2} \frac{\psi_r(l/n)}{l} \ll \frac{1}{nj\sqrt{j}} = \rho_j(n), \quad j \le n.
\end{aligned}
$$

This and (3.13) establish $\bar{b}_2(\bar{y}_r)Q = \bar{\rho}(n)$. □

Proof of Proposition 3.8. Choose

$$\bar{x} = \bar{Y}_r := \bar{y}_r - \bar{b}_1(\bar{y}_r) - \frac{3}{2\lambda_r}\bar{b}_2(\bar{y}_r)$$

and apply Lemmata 3.6, 3.9, and 3.10 to verify that

$$
\begin{aligned}
\bar{Y}_r Q - \lambda_r \bar{Y}_r = &\Big(\lambda_r \bar{y}_r - \lambda_r \bar{b}_1(\bar{y}_r) - \frac{1}{2}\bar{b}_2(\bar{y}_r)\Big) - \bar{b}_2(\bar{y}_r) + \bar{\rho}(n) \\
&- \lambda_r\Big(\bar{y}_r - \bar{b}_1(\bar{y}_r) - \frac{3}{2\lambda_r}\bar{b}_2(\bar{y}_r)\Big) = \bar{\rho}(n).
\end{aligned}
$$

By Lemma 3.1, the eigenvalue μ_r of Q satisfies the inequality

$$|\mu_r - \lambda_r| \cdot ||\bar{Y}_r|| \le ||\bar{Y}_r Q - \lambda_r \bar{Y}_r|| \ll n^{-1}. \tag{3.14}$$

As we have observed such μ_r is unique if n is large enough. It remains to find the lower bound of $||\bar{Y}_r||$. By (3.2), we already have

$$||\bar{y}_r|| = (2r)^{-1/2}(1 + O(1/n)).$$

Furthermore, as seen from (3.3) and (3.4),

$$||\bar{b}_1(\bar{y}_r)|| + ||\bar{b}_2(\bar{y}_r)|| \ll (\log n)/n.$$

Hence,

$$||\bar{Y}_r|| \ge ||\bar{y}_r|| - ||\bar{b}_1(\bar{y}_r)|| - (3/2)r||\bar{b}_2(\bar{y}_r)|| \ge 2^{-1}r^{-1/2}$$

provided that n is sufficiently large and r is fixed.

Now application of (3.14) completes the proof of Proposition 3.8. □

Proof of Theorem 1.2. Proposition 3.8 shows that $\mu_2 = -1/2 + O(n^{-1})$ is the minimal eigenvalue of matrix Q. Thus, $\bar{x}Q\bar{x}' \geq \big(-1/2 + O(n^{-1})\big)||\bar{x}||^2$ for all $\bar{x} \in \mathbb{R}^n$. Consequently, by the definitions,

$$V_n(\bar{a}) \leq D_n(\bar{a}) + \big(1/2 + O(n^{-1})\big)B_n(\bar{a}) \leq \big(3/2 + O(n^{-1})\big)B_n(\bar{a})$$

for all $\bar{a} \in \mathbb{R}^n$ with an absolute constant in the symbol $O(\cdot)$.

On the other hand, we have found the values of the involved quadratic forms at the point $\bar{a} = \bar{a}_2$. Namely, by Lemma 3.5,

$$\begin{aligned} V_n(\bar{a}_2) &= B_n(\bar{a}_2) + O(n^{-1}) - \big(\Delta_{n1}(\bar{a}_2) + \Delta_{n2}(\bar{a}_2)\big) \\ &= \big(3/2 + O(n^{-1})\big)B_n(\bar{a}_2). \end{aligned}$$

This gives the lower bound for the supremum of ratio $V_n(\bar{a})/B_n(\bar{a})$ over $\bar{a} \in \mathbb{R}^n$. $\square$

3.4 Hypothesis for bound improvement

Taking inspiration from J. Lee [27], who sharpened inequality (1.1), we also raise the question whether Theorem 1.1 can be improved in a similar manner. To check this hypothesis, we turn to numerical methods.

The figure 3.1 presents computer calculation results for strongly additive functions defined on random permutations. The sub-figure on the left depicts the behavior of $\max_{\bar{a}\in\mathbb{R}^n} V_n(\bar{a})/D_n(\bar{a})$ and $\max_{\bar{a}\in\mathbb{R}^n} V_n(\bar{a})/B_n(\bar{a})$. The calculations are done by transforming the ratios into $\bar{x}V_D\bar{x}'/||\bar{x}||^2$ ($\bar{x}V_B\bar{x}'/||\bar{x}||^2$ respectively) by the linear variable substitution (here V_D and V_B are the appropriate symmetric $n \times n$ matrices). Finally, the largest eigenvalues of V_D and V_B are graphed.

The sub-figure on the right in figure 3.1 shows three eigenvalues of the matrix Q of the maximal module for differing $n \leq 100$ values. They support the hypothesis that the bound

$$V_n(\bar{a}) \leq \frac{3}{2}D_n(\bar{a}) - \frac{C}{n}, \quad \bar{a} \in \mathbb{R}^n,\ C \in \mathbb{R}_+,\ n \geq 2,$$

is true.

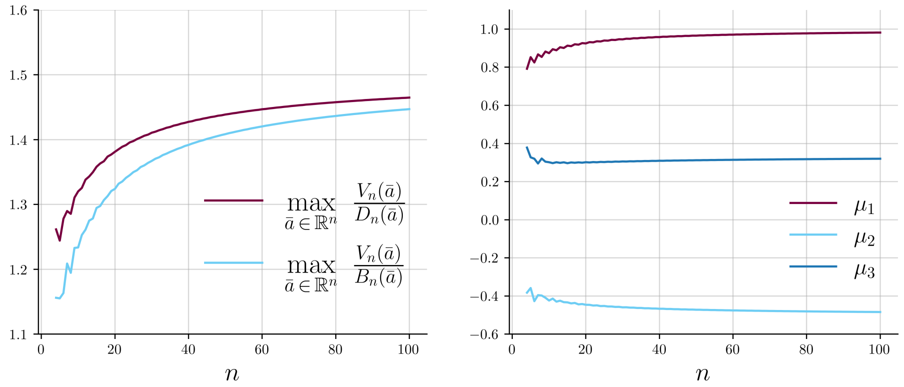

Figure 3.1: Numerical calculations for the variance of strongly additive functions.

Chapter 4

Completely additive functions defined on weighted multisets

This chapter is devoted to variance estimates for completely additive functions defined on weighted multisets. We prove the theorems stated in Section 1.9.2, using the previously introduced notation.

4.1 Auxiliary lemmata

Let $\bar{\gamma}^{(x)} = (\gamma_1^{(x)}, \gamma_2^{(x)}, \dots)$ be an infinite dimensional vector with coordinates that are independent r.vs. following the negative binomial law with parameters $\gamma_j^{(x)} \sim NB(\pi(j), x^j)$, i.e.,

$$\mathbf{P}(\gamma_j^{(x)} = m) = \binom{\pi(j)+m-1}{m}(1-x^j)^{\pi(j)}x^{jm}, \quad m \in \mathbb{N}_0,$$

here $0 < x \le q^{-1}$. Suppose that the structural vector $\bar{k}(\sigma)$ is also expanded to an infinite dimension accordingly $\bar{k}(\sigma) = (k_1(\sigma), k_2(\sigma), \dots, k_n(\sigma), 0, 0, \dots) \in \mathbb{N}_0^\infty$.

Lemma 4.1. *Let* $\Psi : \mathbb{N}_0^\infty \to \mathbb{R}$ *be a mapping, such that* $\mathbf{E}\big|\Psi(\bar{\gamma}^{(x)})\big| < \infty$. *Then*

$$\mathbf{E}\Psi(\bar{\gamma}^{(x)}) = (1-qx)\bigg(\Psi(\bar{0}) + \sum_{n\ge 1}\mathbf{E}_n\Psi(\bar{k}(\varrho))q^n x^n\bigg), \quad 0 < x < q^{-1}.$$

This expression follows from the so-called *conditioning relation* for weighted multisets

$$\nu_n(\bar{k}(\varrho) = \bar{s}) = \mathbf{P}(\bar{\gamma}^{(x)} = \bar{s}|\theta^{(x)} = n),$$

where $\theta^{(x)} = \sum_{j\geq 1} j\gamma_j^{(x)}$. For proof, see [13].

This Lemma allows us to calculate the moments of the coordinates of a random structural vector. It remains to just choose the right mappings Ψ.

Lemma 4.2. *The following formulae for the moments of the random structural vector coordinates are true:*

$$\mathbf{E}_n k_j(\varrho) = \pi(j) \sum_{1\leq k\leq n/j} q^{-jk},$$

$$\mathbf{E}_n\big(k_j(\varrho)(k_j(\varrho)-1)\big) = \pi(j)(\pi(j)+1) \sum_{1\leq k\leq n/j} (k-1)q^{-jk}.$$

Also, if $i \neq j$, then

$$\mathbf{E}_n\big(k_i(\varrho)k_j(\varrho)\big) = \pi(i)\pi(j) \sum_{\substack{l,k\geq 1\\ il+jk\leq n}} q^{-il-jk}.$$

Proof. The proof relies on Lemma 4.1.

1) Let $\Psi(\bar{k}) = k_j$. Then

$$\mathbf{E}\Psi(\bar{\gamma}^{(x)}) = \mathbf{E}\gamma_j^{(x)} = \frac{\pi(j)x^j}{1-x^j} = \pi(j)\sum_{n\geq 0} x^{(n+1)j}.$$

On the right side of the identity in Lemma 4.1, we have

$$(1-qx)\left(\Psi(\bar{0}) + \sum_{n\geq 1} \mathbf{E}_n\Psi(\bar{k}(\varrho))q^n x^n\right) = (1-qx)\sum_{n\geq 1}\mathbf{E}_n k_j(\varrho)q^n x^n.$$

Combining both, we get

$$\pi(j)\sum_{n\geq 0} x^{(n+1)j} = (1-qx)\sum_{n\geq 1}\mathbf{E}_n k_j(\varrho)q^n x^n,$$

$$\left(\pi(j)\sum_{n\geq 0} x^{(n+1)j}\right)\left(\sum_{k\geq 0}(qx)^k\right) = \sum_{n\geq 1}\mathbf{E}_n k_j(\varrho)q^n x^n,$$

$$\pi(j)\sum_{n\geq 1}x^n\sum_{\substack{jk+l=n\\ l\geq 0,\,k\geq 1}}q^l=\sum_{n\geq 1}\mathbf{E}_n k_j(\varrho)q^n x^n.$$

Now compare the coefficients of x^n to obtain

$$\mathbf{E}_n k_j(\varrho)=\pi(j)\sum_{\substack{jk+l=n\\ l\geq 0,\,k\geq 1}}q^{l-n}=\pi(j)\sum_{1\leq k\leq n/j}q^{-jk}.$$

Other expressions are derived accordingly by using different mappings Ψ.

2) Let $\Psi(\bar{k})=k_j(k_j-1)$. Then

$$\begin{aligned}\mathbf{E}\Psi(\bar{\gamma}^{(x)})&=\mathbf{E}\gamma_j^{(x)}(\gamma_j^{(x)}-1)=\mathbf{E}\big(\gamma_j^{(x)}\big)_2\\ &=\left(\frac{x^j}{1-x^j}\right)^2\frac{\Gamma(\pi(j)+2)}{\Gamma(\pi(j))}=\left(\frac{x^j}{1-x^j}\right)^2\pi(j)(\pi(j)+1).\end{aligned}$$

Also,

$$\begin{aligned}&(1-qx)\Bigg(\Psi(\bar{0})+\sum_{n\geq 1}\mathbf{E}_n\Psi(\bar{k}(\varrho))q^n x^n\Bigg)\\ &=(1-qx)\sum_{n\geq 1}\mathbf{E}_n\big(k_j(\varrho)(k_j(\varrho)-1)\big)q^n x^n.\end{aligned}$$

Combining both expressions, we get

$$\left(\frac{x^j}{1-x^j}\right)^2\pi(j)(\pi(j)+1)=(1-qx)\sum_{n\geq 1}\mathbf{E}_n\big(k_j(\varrho)(k_j(\varrho)-1)\big)q^n x^n,$$

$$\pi(j)(\pi(j)+1)\Bigg(\sum_{k\geq 1}x^{kj}\Bigg)^2\sum_{l\geq 0}(qx)^l=\sum_{n\geq 1}\mathbf{E}_n\big(k_j(\varrho)(k_j(\varrho)-1)\big)q^n x^n,$$

$$\pi(j)(\pi(j)+1)\sum_{n\geq 2j}x^n\Bigg(\sum_{\substack{kj+rj+l=n\\ k,r\geq 1,\,l\geq 0}}q^l\Bigg)=\sum_{n\geq 1}\mathbf{E}_n\big(k_j(\varrho)(k_j(\varrho)-1)\big)q^n x^n.$$

Comparing coefficients of x^n

$$\pi(j)(\pi(j)+1)\Bigg(\sum_{\substack{kj+rj+l=n\\ k,r\geq 1,\,l\geq 0}}q^l\Bigg)=\mathbf{E}_n\big(k_j(\varrho)(k_j(\varrho)-1)\big)q^n,$$

$$\pi(j)(\pi(j)+1)\Bigg(\sum_{\substack{1\leq r\leq n/j,\\ 1\leq k\leq (n-r)/j}}q^{-j(k+r)}\Bigg)=\mathbf{E}_n\big(k_j(\varrho)(k_j(\varrho)-1)\big),$$

$$\pi(j)(\pi(j)+1)\Big(\sum_{1\le k\le n/j}(k-1)q^{-jk}\Big)=\mathbf{E}_n\big(k_j(\varrho)(k_j(\varrho)-1)\big).$$

3) Let $\Psi(\bar{k})=k_jk_i$, $i\neq j$. Then

$$\mathbf{E}\Psi(\bar{\gamma}^{(x)})=(1-qx)\Big(\Psi(\bar{0})+\sum_{n\ge1}\mathbf{E}_n\Psi(\bar{k}(\varrho))q^nx^n\Big),$$

$$\mathbf{E}\big(\gamma_j^{(x)}\gamma_i^{(x)}\big)=\frac{\pi(j)x^j\pi(i)x^i}{(1-x^j)(1-x^i)}=(1-qx)\sum_{n\ge1}\mathbf{E}_n\big(k_j(\varrho)k_i(\varrho)\big)q^nx^n,$$

$$\pi(j)\pi(i)\Big(\sum_{k\ge1}x^{ki}\Big)\Big(\sum_{l\ge1}x^{lj}\Big)\Big(\sum_{m\ge0}(qx)^m\Big)=\sum_{n\ge1}\mathbf{E}_n\big(k_j(\varrho)k_i(\varrho)\big)q^nx^n,$$

$$\pi(j)\pi(i)\sum_{n\ge0}x^n\sum_{\substack{ki+lj+m=n,\\ k,l\ge1,\,m\ge0}}q^m=\sum_{n\ge1}\mathbf{E}_n\big(k_j(\varrho)k_i(\varrho)\big)q^nx^n.$$

Hence,

$$\pi(j)\pi(i)\sum_{ki+lj\le n}q^{-ki-lj}=\mathbf{E}_n\big(k_j(\varrho)k_i(\varrho)\big).$$

□

Now we have all the tools needed to derive the variance expression of completely additive functions.

Proof of Theorem 1.4. Using Lemma 4.2, we obtain

$$\mathbf{E}_n(h(\bar{a}))=\sum_{1\le j\le n}a_j\mathbf{E}_nk_j(\varrho)=\sum_{1\le j\le n}a_j\pi(j)\sum_{1\le k\le n/j}q^{-jk}$$

and

$$\begin{aligned}\mathbf{E}_n\big(h(\bar{a})\big)^2&=\mathbf{E}_n\Big(\sum_{1\le j\le n}a_jk_j(\varrho)\Big)^2\\&=\mathbf{E}_n\Big(\sum_{1\le j\le n}a_j^2k_j^2(\varrho)+\sum_{\substack{i,j\le n\\ i\neq j}}a_ia_jk_i(\varrho)k_j(\varrho)\Big)\\&=\sum_{1\le j\le n}a_j^2\big(\mathbf{E}_nk_j(\varrho)+\mathbf{E}_n\big(k_j(\varrho)\big)_2\big)+\sum_{\substack{i,j\le n\\ i\neq j}}a_ia_j\mathbf{E}_n\big(k_i(\varrho)k_j(\varrho)\big).\end{aligned}$$

Finally,

$$
\begin{aligned}
\mathbf{V}_n(h(\bar{a})) = \mathbf{E}_n\big(h(\bar{a})\big)^2 - \big(\mathbf{E}_n h(\bar{a})\big)^2 = \sum_{1\le j\le n} a_j^2\big(\mathbf{E}_n k_j(\varrho) + \mathbf{E}_n\big(k_j(\varrho)\big)_2\big) \\
+ \sum_{\substack{i,j\le n\\ i\ne j}} a_i a_j \mathbf{E}_n\big(k_i(\varrho)k_j(\varrho)\big) - \bigg(\sum_{1\le j\le n} a_j\pi(j) \sum_{1\le k\le n/j} q^{-jk}\bigg)^2 \\
= \sum_{1\le j\le n} a_j^2\bigg(\pi(j)\sum_{1\le k\le n/j} q^{-jk} \\
+ \pi(j)(\pi(j)+1)\sum_{1\le k\le n/j}(k-1)q^{-jk}\bigg) \\
+ \sum_{\substack{i,j\le n\\ i\ne j}} a_i a_j \pi(i)\pi(j) \sum_{\substack{l,k\ge 1\\ il+jk\le n}} q^{-il-jk} \\
- \sum_{i,j\le n} a_i a_j \pi(i)\pi(j) \sum_{\substack{l\le n/i,\\ k\le n/j}} q^{-il-jk} \\
= \sum_{1\le jk\le n} a_j^2 \pi(j) k q^{-jk} + \sum_{1\le jk\le n} a_j^2\pi(j)^2(k-1)q^{-jk} \\
- \sum_{1\le im\le n} a_i^2\pi(i)^2(m-1)q^{-im} \\
- \sum_{\substack{il+jk>n\\ il\le n,\, jk\le n}} a_i a_j \pi(i)\pi(j) q^{-il-jk} \\
= \sum_{1\le jk\le n} a_j^2\pi(j)kq^{-jk} - \sum_{\substack{il+jk>n\\ il\le n,\, jk\le n}} a_i a_j\pi(i)\pi(j)q^{-il-jk}.
\end{aligned}
$$

We have obtained the expression (1.9). □

4.2 Variance bounds for completely additive functions

In this section, we find the variance estimate of completely additive functions. The main goal is to bind the variance $\mathbf{V}_n$ by its first term in (1.9), which will be denoted by $\mathbf{B}_n$

$$
\mathbf{B}_n(\bar{a}) := \sum_{1\le jk\le n} a_j^2\pi(j)kq^{-jk}.
$$

The second term will be estimated using $\mathbf{R}_n$

$$\mathbf{R}_n(\bar{a}) := \sum_{m\le n} mq^{-2m}\Big(\sum_{j|m} a_j\pi(j)\Big)^2.$$

Proof of Theorem 1.5. Inequality (1.10) is equivalent to

$$-\sum_{\substack{il+jk>n\\ il\le n,\, jk\le n}} a_i a_j \pi(i)\pi(j) q^{-il-jk} \le \frac{1}{2}\mathbf{R}_n(\bar{a}). \tag{4.1}$$

To prove this, we group the summands on the left-hand side

$$\begin{aligned}
&-\sum_{\substack{il+jk>n\\ il\le n,\, jk\le n}} a_i a_j \pi(i)\pi(j) q^{-il-jk}\\
&= -\sum_{\substack{m_1+m_2>n\\ m_1,m_2\le n}} \Big(q^{-m_1}\sum_{i|m_1} a_i\pi(i)\Big)\Big(q^{-m_2}\sum_{j|m_2} a_j\pi(j)\Big).
\end{aligned}$$

Now, it remains to apply inequality (2.1), with

$$b_m = q^{-m}\sum_{j|m} a_j\pi(j), \quad m\le n.$$

The claim (4.1) is proved. By Lemma 2.4, inequality (4.1) becomes an equality if

$$q^{-m}\sum_{j|m} a_j\pi(j) = 3m-2n-1.$$

From this, we find a_j by using Möbius inversion formula. So,

$$a_j = a_j^* = \frac{3}{\pi(j)}\sum_{d|j} dq^d\mu\Big(\frac{j}{d}\Big) - \frac{2n+1}{\pi(j)}\sum_{d|j} q^d\mu\Big(\frac{j}{d}\Big).$$

It remains to apply (1.8).

□

Subsequently, we will use the following notation

$$\bar{j} := (1,2,3,\dots,n) \in \mathbb{R}^n.$$

Proof of Corollary 1.6. To prove the left inequality in (1.12), it is sufficient to show that $\mathbf{R}_n(\bar{a}) \le \mathbf{B}_n(\bar{a})$, $n \ge 2$ and $\bar{a} \ne \bar{0}$. We estimate the inner sum of $\mathbf{R}_n$ as follows

$$\Big(\sum_{j|m} a_j \pi(j)\Big)^2 \le \sum_{j|m} \frac{a_j^2 \pi(j)}{j} \cdot \sum_{j|m} j\pi(j) = \sum_{j|m} \frac{a_j^2\pi(j)}{j} \cdot q^m; \tag{4.2}$$

therefore,

$$\begin{aligned} \mathbf{R}_n(\bar{a}) &= \sum_{m\le n} mq^{-2m}\Big(\sum_{j|m} a_j\pi(j)\Big)^2 \\ &\le \sum_{m\le n} mq^{-m} \sum_{j|m} \frac{a_j^2\pi(j)}{j} = \mathbf{B}_n(\bar{a}). \end{aligned} \tag{4.3}$$

Actually, we can change inequalities (4.2) and (4.3) to be strict, since the Cauchy inequality is strict except for the case when $\bar{a}$ is proportional to $\bar{j}$. To tackle the second inequality of (1.12), we find the exact expression

$$\mathbf{V}Y_n = \sum_{1\le j\le n} a_j^2 \mathbf{V}\gamma_j = \sum_{1\le j\le n} \frac{a_j^2\pi(j)q^j}{(q^j-1)^2} = \sum_{1\le j\le n} a_j^2\pi(j)\sum_{k\ge 1} kq^{-jk}.$$

Further, we use an elementary formula

$$\sum_{k\ge n} kx^{-k} = \frac{x^{1-n}(nx-n+1)}{(x-1)^2}, \quad |x|>1,\, n\in\mathbb{N}. \tag{4.4}$$

It remains to estimate

$$\begin{aligned} \mathbf{V}Y_n - \mathbf{B}_n(\bar{a}) &= \sum_{j\le n} a_j^2\pi(j) \sum_{k\ge\lfloor n/j\rfloor+1} kq^{-jk} \\ &= \sum_{j\le n} a_j^2\pi(j)\Big(\frac{q^{-j\lfloor n/j\rfloor}(q^j(\lfloor n/j\rfloor+1)-\lfloor n/j\rfloor)}{(q^j-1)^2}\Big) \\ &\ge q^{-n}\sum_{j\le n} a_j^2\pi(j)\frac{q^j(n/j)-(n/j)}{(q^j-1)^2} \\ &\ge q^{-n}\sum_{j\le n} a_j^2\pi(j)\frac{q^j}{(q^j-1)^2}\frac{q^j-1}{q^j} \ge q^{-n}\frac{q-1}{q}\mathbf{V}Y_n. \end{aligned}$$

□

Lemma 4.3. *The quadratic forms $\mathbf{B}_n(\bar{a}-t\bar{j})$ and $\mathbf{R}_n(\bar{a}-t\bar{j})$ attain their minimum values with respect to $t\in\mathbb{R}$ when*

$$t = t_{\bar{a}} := \frac{2}{(n+1)n}\sum_{m\le n} mq^{-m}\sum_{j|m} a_j\pi(j).$$

Proof. We begin with

$$\begin{aligned}\mathbf{B}_n(\bar{a} - t_{\bar{a}}\bar{j}) &= \sum_{1 \le jk \le n} (a_j - t_{\bar{a}} j)^2 \pi(j) k q^{-jk} \\ &= \mathbf{B}_n(\bar{a}) - tn(n+1)t_{\bar{a}} + t^2 \frac{n(n+1)}{2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{R}_n(\bar{a} - t\bar{j}) &= \sum_{1 \le m \le n} m q^{-2m} \Big(\sum_{j|m} (a_j - t_{\bar{a}} j)\pi(j) \Big)^2 \\ &= \mathbf{R}_n(\bar{a}) - tn(n+1)t_{\bar{a}} + t^2 \frac{n(n+1)}{2}.\end{aligned}$$

By using properties of parabolas, we can deduce that $\mathbf{B}_n(\bar{a} - t\bar{j})$ and $\mathbf{R}_n(\bar{a} - t\bar{j})$ attain their minimal values when $t = t_{\bar{a}}$. □

Proof of Theorem 1.7. First, notice

$$\mathbf{V}_n h(\bar{a}) = \mathbf{V}_n\big(h(\bar{a}) - tn\big) = \mathbf{V}_n h(\bar{a} - t\bar{j}), \quad \forall t \in \mathbb{R}. \tag{4.5}$$

Hence, the right (1.13) inequality follows from (1.10) and (4.5). The left-hand side can be derived from (2.2). We only need to apply the substitution

$$\tilde{b}_m = q^{-m} \sum_{j|m} (a_j - t_{\bar{a}} j)\pi(j),$$

$m \le n$ to the quadratic form $\mathbf{R}_n$. It remains to show that the condition $\sum_{m \le n} m\tilde{b}_m = 0$ holds. Indeed

$$\begin{aligned}\sum_{m \le n} m\tilde{b}_m &= \sum_{m \le n} m q^{-m} \sum_{j|m} (a_j - t_{\bar{a}} j)\pi(j) \\ &= \frac{n(n+1)}{2} t_{\bar{a}} - t_{\bar{a}} \sum_{m \le n} m q^{-m} q^m = 0.\end{aligned}$$

□

Proof of Corollary 1.8. The result follows from $\mathbf{B}_n \ge \mathbf{R}_n$ and (1.13). □

4.3 Asymptotic expressions

The goal of this section is to derive asymptotic formulas for the supremum and infimum of $\frac{\mathbf{V}_n h(\bar{a})}{\mathbf{B}_n(\bar{a})}$. Besides the already introduced notation, we will also use Θ to represent any real value whose absolute value does not exceed 1, i.e., instead of $|A| \le |B|$, we will write $A = \Theta B$. We begin this section with some auxiliary lemmata.

Lemma 4.4. *Let a_j^* be as defined in (1.11). If $1 \le j \le n$, then*

$$\begin{aligned} |\pi(j)a_j^* - 3jq^j + (2n+1)j\pi(j)| &\le 3jq^{j/2}, \\ \pi(j)a_j^* &= \Theta(6jq^j + (2n+1)q^j), && (4.6) \\ a_j^* &= \Theta(6j + (2n+1))4j. && (4.7) \end{aligned}$$

Proof. Starting from the definition of a_j^*, we obtain

$$\begin{aligned} |\pi(j)a_j^* - 3jq^j + (2n+1)j\pi(j)| &= \Big|3 \sum_{\substack{d|j \\ d \le j/2}} dq^d \mu(j/d)\Big| \\ &\le 3 \sum_{\substack{d|j \\ d \le j/2}} dq^d \le 3 \sum_{1 \le d \le j/2} dq^d \le 3\frac{j}{2} \sum_{1 \le d \le j/2} q^d \\ &= \frac{3jq}{2}\frac{q^{j/2}-1}{q-1} \le 3jq^{j/2}. \end{aligned}$$

Inequalities (4.6) and (4.7) can be derived using $\frac{q^j}{4} \le j\pi(j) < q^j$

$$\begin{aligned} \pi(j)a_j^* &= \Theta(3jq^j + (2n+1)j\pi(j) + 3jq^{j/2}) = \Theta(6jq^j + (2n+1)q^j), \\ a_j^* &= \Theta(6j + 2n + 1)q^j \pi(j)^{-1} = \Theta(6j + 2n + 1)4j. \end{aligned}$$

□

Lemma 4.5. *If $n \ge 2$, we have*

$$\sum_{k \ge 2} kq^{-jk} \le 8q^{-2j}.$$

Proof. We will use the identity (4.4)

$$\sum_{k \ge 2} kq^{-jk} = \frac{q^{-j}(2q^j - 2 + 1)}{(q^j - 1)^2} = \frac{2 - q^{-j}}{(q^j-1)^2} \le \frac{2}{q^{2j}}\left(\frac{q^j}{q^j - 1}\right)^2 \le 8q^{-2j}.$$

□

Lemma 4.6. *If $n \geq 2$, then*

$$\sum_{\substack{1 \leq kj \leq n \\ k \geq 2}} (a_j^*)^2 \pi(j) k q^{-jk} = O(n^2).$$

Proof. First, we use the previously proven Lemma 4.5 to obtain

$$\begin{aligned}
\sum_{\substack{1 \leq kj \leq n \\ k \geq 2}} (a_j^*)^2 \pi(j) k q^{-jk} &\leq \sum_{1 \leq j \leq n} (a_j^*)^2 \pi(j) \sum_{k \geq 2} k q^{-jk} \\
&\leq 8 \sum_{1 \leq j \leq n} (a_j^*)^2 \pi(j) q^{-2j} \leq 8 \sum_{1 \leq j \leq n} (a_j^*)^2 q^{-j} j^{-1}.
\end{aligned}$$

Next, inequality (4.7) yields

$$\begin{aligned}
8 \sum_{1 \leq j \leq n} (a_j^*)^2 q^{-j} j^{-1} &= \Theta \sum_{1 \leq j \leq n} (6j + 2n + 1)^2 q^{-j} j \\
&= \Theta \sum_{1 \leq j \leq n} j^3 q^{-j} + \Theta \sum_{1 \leq j \leq n} j^2 q^{-j} (2n+1) \\
&\quad + \Theta \sum_{1 \leq j \leq n} j q^{-j} (2n+1)^2 \\
&= O(n^2).
\end{aligned}$$

□

Lemma 4.7. *If $n \geq 2$, then*

$$\mathbf{R}_n(\bar{a}^*) = \frac{(n-1)n(n+1)(n+2)}{4}.$$

Proof. We only need to use the definitions of $\mathbf{R}_n$ and $\bar{a^*}$. Thus,

$$\begin{aligned}
\mathbf{R}_n(\bar{a}^*) &= \sum_{1 \leq m \leq n} m q^{-2m} (q^m (3m - 2n - 1))^2 \\
&= \sum_{1 \leq m \leq n} 9m^3 - 6m^2(2n+1) + m(2n+1)^2 \\
&= \frac{9}{4} n^2 (n+1)^2 - 6(2n+1) \frac{n(n+1)(2n+1)}{6} \\
&\quad + \frac{n(n+1)}{2} (2n+1)^2 \\
&= \frac{(n-1)n(n+1)(n+2)}{4}.
\end{aligned}$$

□

Lemma 4.8. *If $q \geq 2$, then*

$$\pi(j)^{-1} = jq^{-j}(1 + \Theta j q^{-j/2}).$$

Proof. We use the property $q^j/j - \pi(j) \leq q^{j/2}$ and obtain

$$\begin{aligned}
\frac{q^j}{j} - \pi(j) &= \Theta q^{j/2}, \\
\frac{q^j}{j\pi(j)} &= 1 + \Theta \frac{q^{j/2}}{\pi(j)}, \\
\pi(j)^{-1} &= jq^{-j}(1 + \Theta j q^{-j/2}).
\end{aligned}$$

□

Lemma 4.9. *If $n \geq 2$, then*

$$\mathbf{B}_n(\bar{a}^*) = \mathbf{R}_n(\bar{a}^*) + O(n^2).$$

Proof. We estimate $\mathbf{B}_n(\bar{a}^*)$ using the results from the lemmata in this section. Namely,

$$\begin{aligned}
\mathbf{B}_n(\bar{a}^*) &= \sum_{1 \leq jk \leq n} a_j^2 \pi(j) k q^{-jk} \\
&= \sum_{1 \leq j \leq n} (a_j^*)^2 \pi(j) q^{-j} + \sum_{\substack{1 \leq kj \leq n \\ k \geq 2}} (a_j^*)^2 \pi(j) k q^{-jk} \\
&= \sum_{1 \leq j \leq n} a_j^* \pi(j) a_j^* q^{-j} + O(n^2) \\
&= \sum_{1 \leq j \leq n} \left(3jq^j - (2n+1) j\pi(j) + O(jq^{j/2}) \right) \\
&\quad \cdot \left(3jq^j \pi(j)^{-1} - (2n+1)j + O(jq^{j/2}\pi(j)^{-1}) \right) + O(n^2) \\
&= \sum_{1 \leq j \leq n} \Big(9j^2 q^{2j} \pi(j)^{-1} - 3(2n+1) j^2 q^j \\
&\quad - 3(2n+1) j^2 q^j + (2n+1)^2 j^2 \pi(j) \\
&\quad + O(j^2 q^{j/2} \pi(j)^{-1} + nj^3 \pi(j) q^{-j/2} + j^3) \Big) q^{-j} + O(n^2) \\
&= \sum_{1 \leq j \leq n} \Big(9j^3 (1 + \Theta j q^{-j/2}) - 6(2n+1) j^2
\end{aligned}$$

$$
\begin{aligned}
&+ (2n+1)^2 j\Big) + O(n^2)\\
&= \mathbf{R}_n(\bar{a}^*) + O(n^2).
\end{aligned}
$$

□

Proof of Theorem 1.9. Inequality follows from Lemma 4.9:

$$\sup_{\bar{a}\in\mathbb{R}^n\setminus\{\bar{0}\}} \frac{\mathbf{V}_n h(\bar{a})}{\mathbf{B}_n(\bar{a})} \geq \frac{\mathbf{V}_n h(\bar{a}^*)}{\mathbf{B}_n(\bar{a}^*)} = 1 + \frac{1}{2}\frac{\mathbf{R}_n(\bar{a}^*)}{\mathbf{B}_n(\bar{a}^*)} \geq \frac{3}{2} - \frac{C}{n^2}.$$

□

An asymptotic formula for the infimum will be proven in the same way. We examine the case of $\bar{a} = \bar{a}_3^*$, where

$$a_{3j}^* = (3n^2+3n+2)j + \frac{1}{\pi(j)}\sum_{d|j}\mu(j/d)q^d(10d^2 - 6(2n+1)d),$$

or alternatively,

$$q^{-j}\sum_{d|j} a_{3d}^*\pi(d) = 10j^2 - 6(2n+1)j + 3n^2 + 3n + 2.$$

Lemma 4.10. *If $n \geq 3$, then*

$$\mathbf{R}_n(\bar{a}_3^*) = \frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{6}.$$

Proof. The proof is a straightforward calculation derived from the definitions of $\mathbf{R}_n$ and $\bar{a}_3$:

$$
\begin{aligned}
\mathbf{R}_n(\bar{a}_3^*) &= \sum_{1\leq m\leq n} mq^{-2m}\big(q^m[10m^2 - 6(2n+1)m + 3n^2+3n+2]\big)^2\\
&= \sum_{1\leq m\leq n} 100m^5 - 120(2n+1)m^4\\
&\quad + (36(2n+1)^2 + 20(3n^2+3n+2))m^3\\
&\quad - 12(2n+1)(3n^2+3n+2)m^2 + (3n^2+3n+2)^2 m\\
&= \frac{100}{12}n^2(n+1)^2(2n^2+2n-1)\\
&\quad - \frac{120}{30}(2n+1)^2 n(n+1)(3n^2+3n-1)\\
&\quad + \frac{1}{4}n^2(n+1)^2\big(36(2n+1)^2 + 20(3n^2+3n+2)\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{12}{6}(2n+1)^2 n(n+1)(3n^2+3n+2) \\
&+ \frac{1}{2} n(n+1)(3n^2+3n+2)^2 \\
= &\ \frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{6}.
\end{aligned}
$$

□

Lemma 4.11. *If* $1 \le j \le n$, $n \ge 3$, *then*

$$
\begin{aligned}
&|\pi(j)a^*_{3j} - (3n^2+3n+2)j\pi(j) - 10j^2q^j + 6(2n+1)jq^j| \\
&\qquad \le (5j^2 + 6j(2n+1))q^{j/2},
\end{aligned}
$$

$$
\begin{aligned}
\pi(j)a^*_{3j} &= \Theta q^j \big((3n^2+3n+2) + 15j^2 + 12j(2n+1)\big), \\
a^*_{3j} &= \Theta\big((3n^2+3n+2) + 15j^2 + 12j(2n+1)\big)4j. \qquad (4.8)
\end{aligned}
$$

Proof. Similarly to Lemma 4.4, we have

$$
\begin{aligned}
&|\pi(j)a^*_{3j} - (3n^2+3n+2)j\pi(j) - 10j^2q^j + 6(2n+1)jq^j| \\
&\quad = \Big| \sum_{\substack{d|j \\ d \le j/2}} (10d^2 - 6(2n+1)d)q^d \mu(j/d) \Big| \\
&\quad \le \sum_{\substack{d|j \\ d \le j/2}} (10d^2 - 6(2n+1)d)q^d \\
&\quad \le 10 \sum_{1 \le d \le j/2} d^2 q^d + 6(2n+1) \sum_{1 \le d \le j/2} dq^d \\
&\quad \le \left(\frac{10}{4}j^2 + \frac{6}{2}(2n+1)j\right) \sum_{1 \le d \le j/2} q^d \\
&\quad \le \left(\frac{5}{2}j^2 + 3(2n+1)j\right) \frac{q(q^{j/2}-1)}{q-1} \le \big(5j^2 + 6(2n+1)j\big)q^{j/2}.
\end{aligned}
$$

□

Lemma 4.12. *If* $n \ge 3$, *then*

$$
\sum_{\substack{1 \le kj \le n \\ k \ge 2}} (a^*_{3j})^2 \pi(j) k q^{-jk} = O(n^4).
$$

Proof. We use the result of Lemma 4.5 and the equality (4.8) as follows:

$$\sum_{\substack{1\le kj\le n\\ k\ge 2}} (a^*_{3j})^2\pi(j)kq^{-jk} \le 8\sum_{1\le j\le n}(a^*_{3j})^2 q^{-j}j^{-1}$$
$$= \Theta\sum_{1\le j\le n}\Big(((3n^2+3n+2)+15j^2+12j(2n+1))4j\Big)^2 q^{-j}j$$
$$\le C\Theta n^4\sum_{1\le j\le n} q^{-j}j^3 = O(n^4).$$

□

Lemma 4.13. *If $n\ge 3$, then*

$$\mathbf{B}_n(\bar a^*_3) = \mathbf{R}_n(\bar a^*_3) + O(n^4). \tag{4.9}$$

Proof. We use the results from the lemmata in this section and obtain

$$\begin{aligned}
\mathbf{B}_n(\bar a^*_3) &= \sum_{1\le jk\le n}(a^*_{3j})^2\pi(j)kq^{-jk}\\
&= \sum_{1\le j\le n}(a^*_{3j})^2\pi(j)q^{-j} + \sum_{\substack{1\le kj\le n\\ k\ge 2}}(a^*_{3j})^2\pi(j)kq^{-jk}\\
&= \sum_{1\le j\le n}(a^*_{3j}\pi(j))a^*_{3j}q^{-j} + O(n^4)\\
&= \sum_{1\le j\le n} j\big((3n^2+3n+2)+10j^2-6(2n+1)j\big)^2 + O(n^4)\\
&= \mathbf{R}_n(\bar a^*_3) + O(n^4).
\end{aligned}$$

□

Lemma 4.14. *If $n\ge 3$, then*

$$t_{\bar a^*_3} = 0.$$

Proof. We use the definition of $t_{\bar a}$

$$\begin{aligned}
\frac{t_{\bar a^*_3}n(n+1)}{2} &= \sum_{m\le n} mq^{-m}\sum_{j|m} a^*_{3j}\pi(j)\\
&= \sum_{m\le n} m(10m^2-6(2n+1)m+(3n^2+3n+2))\\
&= \frac{10}{4}n^2(n+1)^2 - \frac{6(2n+1)^2n(n+1)}{6}
\end{aligned}$$

$$+\frac{n(n+1)(3n^2+3n+2)}{2}$$
$$=0.$$

□

Proof of Theorem 1.10. Inequality follows from (4.9) and (1.13):

$$\begin{aligned}
\inf_{\bar{a}\in\mathbb{R}^n\setminus\mathbb{R}(\bar{\jmath})} \frac{\mathbf{V}_n h(\bar{a})}{\mathbf{B}_n(\bar{a})} \le \frac{\mathbf{V}_n(\bar{a}_3^*)}{\mathbf{B}_n(\bar{a}_3^*)} &= 1-\frac{1}{3}\frac{\mathbf{R}_n(\bar{a}_3^*)}{\mathbf{B}_n(\bar{a}_3^*)}\\
&= 1-\frac{1}{3}\frac{\mathbf{R}_n(\bar{a}_3^*)}{\mathbf{R}_n(\bar{a}_3^*)+O(n^4)}\\
&\le \frac{2}{3}+\frac{C}{n^2}.
\end{aligned}$$

The choice of $\bar{a}_3^*$ implies that on the left side of (1.13) we have an equality.

□

Chapter 5

Conclusions

- Constant $3/2$ remains optimal for large classes of additive functions in their respective analogs of Turán–Kubilius inequality.
- Results from probabilistic number theory are expected to have analogs for decomposable combinatorial structures.
- Techniques used to establish probabilistic results in number theory are useful for the analysis of analogous problems for decomposable combinatorial structures.

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Santrauka (Summary in Lithuanian)

Tyrimo objektas

Disertacijoje tiriamos adityviųjų funkcijų virš atsitiktinių keitinių ir svorinių multiaibių dispersijos bei jų įverčiai.

Aktualumas

Keitiniai užimą centrinę vietą kombinatorikoje. Jie natūraliai taikomi sprendžiant įvairius baigtinės aibės rikiavimo uždavinius. Algebriniu požiūriu, keitiniai sudaro simetrines grupes, padedančias atskleisti tiek geometrines, tiek algebrines simetrijas. Kaip paprasčiausi iš žymėtų orentuotųjų grafų, jie tampa pradiniu tyrimų lauku siekiant plačių apibendrinimų sudėtingesnių grafų klasėse. Įvairiai interpretuojami, keitiniai yra svarbūs daugelyje matematinių ir fizikinių tyrimų.

Keitinių savybės dažniausiai išreiškiamos jų aibėje apibrėžtais atvaizdžiais, iš kurių savo svarba išsiskiria adityviosios funkcijos. Jomis užkoduojamos įvairios keitinių charakteristikos, tokios kaip keitinio eilė grupėje ([9], [51]) ir jo ciklų ilgių specifinės savybės (žr. pvz., [49]). Be to, adityviosios funkcijos aptinkamos ir fizikiniuose modeliuose, pvz., Hamiltonianuose, apibrėžiamuose Bose dujų teorijoje (žr. [5] ir kitus ten cituojamus straipsnius). Kai keitinys imamas atsitiktinai, adityviųjų funkcijų elgsena yra nenuspėjama. Taikydami reikšmių vidurkius, turime gebėti įvertinti jų dispersijas. Pastarųjų įverčiai naudojami įrodinėjant ribines teoremas, didžiųjų skaičių dėsnius. Antrieji ir bendresni laipsniniai momentai naudojami apibrėžiant funkcijų erdves. Todėl pagrindinis pirmosios disertacijos dalies tikslas – maksimaliai tikslūs adityviųjų funkcijų dispersijų įverčiai.

Antroje disertacijos dalyje panaši metodologija plėtojama adityviosioms funkcijoms, apibrėžtoms virš svorinių multiaibių. Visų pirma, šios kombinatorinės struktūros apima moninius polinomus virš baigtinio kūno, sudarančius vadinamuosius adityviuosius pusgrupius. Tai ypač populiari ir svarbi teorinei informatikai bei kriptografijai struktūrų klasė, žr. [6, Chapter 2]. Adityviosiomis funkcijomis, apibrėžtomis šiuose pusgrupiuose, galima aproksimuoti polinomo skaidymo kūno laipsnį (žr. pvz., [7], [51], [50, Chapter 2], [42]). Jos išreiškia neredukuojamų polinomo daugiklių savybes. Disertacijoje nagrinėjama, kiek bendresnė negu polinomai virš baigtinio kūno struktūrų klasė – vėrinių multiaibės. Jas galima interpretuoti, kaip cikliškai ekvivalenčius žodžius, sudarytus iš baigtinės abėcėlės raidžių [41]. Kaip ir keitinių atveju, bendros adityviųjų funkcijų savybės padeda atskleisti svorinių multiaibių charakteristikas. Kai šios struktūros iš savosios klasės imamos atsitiktinai, tokių funkcijų vidurkių ir dispersijų vertinimo problema tampa aktualia.

Tikslai ir uždavinys

Pagrindinis šio darbo tikslas – tirti adityviųjų funkcijų, apibrėžtų virš skaidžiųjų kombinatorinių struktūrų, dispersijos asimptotiką. Keliamos šios užduotys:

- Stipriai adityvioms funkcijoms, apibrėžtoms virš atsitiktinių keitinių, gauti dispersijos įverčius su tiksliomis asimptotinėmis konstantomis ir kaip įmanoma geresniu liekamuoju nariu.
- Visiškai adityvioms funkcijoms, apibrėžtoms virš atsitiktinių svorinių multiaibių, gauti dispersijos įverčius su tiksliomis asimptotinėmis konstantomis ir kaip įmanoma geresniu liekamuoju nariu.

Metodai

Disertacijoje taikomi skaičiuojamosios ir tikimybinės kombinatorikos, skaičių teorijos, tiesinės algebros bei skaitinės analizės metodai. Naudojami J. Kubiliaus [26] ir J. Lee [27] darbuose pristatyti kovariacijos matricos analizės būdai, paremti ortogonaliųjų polinomų savybėmis. Kai kurios kvadratinės formos, atsirandančios adityviųjų funkcijų dispersijos išraiškose, yra vertinamos pasinaudojant J. Klimavičiaus ir E. Manstavičiaus rezultatais [19]. Svarbiausias dispersijos įverčio pagerinimas yra pasiektas panaudojus vadinamąjį matricos laipsnių algoritmą, skirtą didžiausiai matricos tikrinei reikšmei rasti [44].

Naujumas

Visi šioje disertacijoje pristatomi rezultatai yra nauji. Jie praplečia adityviųjų funkcijų, apibrėžtų virš atsitiktinių kombinatorinių objektų, teoriją.

Aprobacija

Tyrimų rezultatai pristatyti šiose konferencijose:

- International Conference on Probability Theory and Number Theory, Palanga (Lithuania), 2024, *Variance of a Strongly Additive Function Defined on Random Permutations.*
- 2025 International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2025), Toronto (Canada), 2025, *On Covariance Matrix Related to Components of a Random Combinatorial Structure.*
- 66^{th} Conference of Lithuanian Mathematical Society, Klaipėda (Lithuania), 2025, *Keturiasdešimtmetės Kubiliaus nelygybės analogas keitiniams* (pristatyta bendraautoriaus E. Manstavičiaus).

Publikacijos

- A. Karbonskis, E. Manstavičius, Effective bounds of the variance of statistics on multisets of necklaces, *Lietuvos matem. rink.*, 61(A):7–12, 2021.
- A. Karbonskis, E. Manstavičius, Variance of a strongly additive function defined on random permutations, *Lithuanian Mathematical Journal*, 61(3):1–13, 2024.
- A. Karbonskis, E. Manstavičius, Variance of statistics defined on random permutations, *Lithuanian Mathematical Journal*, 65(4):548–563, 2025.

Išvados

- Dispersijos viršutiniuose įverčiuose esanti konstanta $3/2$ išlieka optimali Turáno–Kubiliaus nelygybės versijose, gautose plačioms kombinatorinių struktūrų klasėms.
- Tikimybinės skaičių teorijos rezultatų analogų galima tikėtis nagrinėjant skaidžiąsias kombinatorines struktūras.
- Tikimybinės skaičių teorijos metodai yra naudingi sprendžiant atitinkamus skaidžiųjų kombinatorinių struktūrų uždavinius.

Istorinė apžvalga

Disertacijoje vertiname priklausomų atsitiktinių dydžių (toliau a.d.), atsirandančių kombinatoriškai apibrėžtose tikimybinėse erdvėse, sumos dispersiją. Tokių uždavinių nagrinėjimą įkvėpė Turáno–Kubiliaus nelygybės sėkmė adityviosioms skaičių teorijos funkcijoms $f\colon \mathbb{N} \to \mathbb{R}$. Pagal apibrėžimą, jos yra išskaidomos taip:

$$f(m) = \sum_{p|m} f\big(p^{\alpha_p(m)}\big) \quad \text{if} \quad m = \prod_{p|m} p^{\alpha_p(m)},$$

čia p yra pirminis skaičius, $\alpha_p(m) \in \mathbb{N}$, o $p|m$ žymime "p dalija m". Jei $f\big(p^k\big) = f(p)$ kiekvienam pirminiam p ir $k \in \mathbb{N}$, tuomet f vadinsime *stipriai adityviąja funkcija.* Šios funkcijos tenkina Turáno–Kubiliaus nelygybę

$$\frac{1}{n}\sum_{m=1}^{n}\Big(\sum_{p|m} f(p) - \sum_{p\le n}\frac{f(p)}{p}\Big)^2 =: K_n(f)\sum_{p\le n}\frac{f(p)^2}{p}, \quad K_n(f)\le C, \tag{5.1}$$

čia $C > 0$ – absoliuti konstanta. Pirmąją tokio tipo nelygybę aprėžtoms $f(p)$ suformulavo ir panaudojo P. Turánas [46]. Joje $K_n(f)$ negalėjo būti aprėžta konstanta, nes dešiniojoje nelygybės pusėje autorius naudojo kitokią negu viršuje, būtent, pirmųjų a.d. momentų sumą. Pateiktąją nelygybę (5.1) įrodė J. Kubilius [23]. Dar 1980-aisiais metais buvo iškelta hipotezė [24], kad konstanta $C = \frac{3}{2}$ yra nepagerinama visiems $n \ge 2$. Vėliau J. Kubilius [26] įrodė, jog

$$\sup_{f\ne 0} K_n(f) = \frac{3}{2} + O\Big(\frac{1}{\log n}\Big),$$

čia konstanta simbolyje $O(\cdot)$, išreiškiančiame aproksimacijos eilę, yra absoliuti. Toliau be $O(\cdot)$ vartosime ir jam ekvivalentų žymenį $\ll$.

J. Lee [27, 28] šį rezultatą dar patikslino gavęs

$$K_n(f) \le \frac{3}{2} - \frac{C_1}{\log n},$$

su tam tikra $C_1 > 0$, kai n yra pakankamai didelis. Mažoms n reikšmėms toks rezultatas iki šiol nėra gautas. Nors J. Lee disertacijoje [27] apžvelgiama šio uždavinio istorija, tačiau svarbus J. Kubiliaus [26] rezultatas yra praleistas ir nepaminėtas.

Turáno–Kubiliaus nelygybė – tai efektyvus atsitiktinių dydžių $f(p)\mathbf{1}\{\alpha_p(m)\ge 1\}$, $p \le n$, sumos dispersijos įvertis, kai m traukiamas iš aibės $\mathbb{N}_n := \{1,\dots,n\}$ su vienoda tikimybe. Čia ir toliau indikatoriaus funkciją žymėsime $\mathbf{1}\{\cdot\}$. Fiksuotam pirminiui p a.d. $\mathbf{1}\{\alpha_p(m)\ge 1\}$ konverguoja į Bernulio a.d., kurio skirstinio parametras yra $1/p$. Tad dešinioji nelygybės (5.1) pusė yra jų antrųjų momentų suma. Konstantos $3/2$ (o ne 1) pasirodymas yra įvykių $p_1|m$ ir $p_2|m$ skirtingiems $p_1, p_2 \le n$ priklausomumo kaina. Platesnę Turáno–Kubiliaus nelygbės apžvalgą galima rasti P.D.T.A. Elliott [8] ir G. Tenenbaum [45] knygose. Šiais laikas Turáno–Kubiliaus nelygybė yra pacituota jau šimtus kartų.

Kombinatoriniai objektai – keitiniai ir polinomai virš baigtinio kūno ar, bendriau, ansambliai ir svorinės multiaibės – panašiai kaip natūralieji skaičiai, kurie skaidomi pirminiais skaičiais, taip pat yra atitinkamų paprastesnių pirminių struktūrų (komponenčių) rinkiniai. Imant juos atsitiktinai, pasireiškia komponenčių priklausomumas, būdingas atsitiktinio natūraliojo skaičiaus pirminių daugiklių priklausomumui. Pasinaudojant analogija su tikimybine skaičių teorija, plėtojama adityviųjų funkcijų, apibrėžtų atsitiktinėse struktūrose, teorija – ją apžvelgsime atsitiktinių keitinių atveju.

Keitinys σ iš simetrinės grupės $\mathbb{S}_n$ vieninteliu būdu skaidomas į nesikertančius ciklus

$$\sigma = \mathfrak{c}_1 \cdots \mathfrak{c}_w, \tag{5.2}$$

čia $w = w(\sigma)$ yra keitinio σ ciklų skaičius. Keitinį σ taip pat galima interpretuoti kaip žymėtąjį digrafą G_σ su viršūnių aibe $V(\sigma) = \mathbb{N}_n$. Šio grafo komponentės, tai orientuotieji ciklai $\mathfrak{c}$ su viršūnių aibėmis $\{k_1, \dots, k_j\} \subset \mathbb{N}_n$, atvaizduojantys

$$k_1 \xrightarrow{\sigma} k_2 \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} k_j \xrightarrow{\sigma} k_1,$$

čia j – ciklo $\mathfrak{c}$ ilgis. Sekdami skaičių teorija, konstruojame atvaizdžius $h\colon \mathbb{S}_n \to \mathbb{R}$, kurių reikšmės priklauso nuo *ciklų struktūros (struktūrinio) vektoriaus*:

$$\bar{\kappa}(\sigma) := \big(\kappa_1(\sigma), \dots, \kappa_n(\sigma)\big),$$

čia $\kappa_j(\sigma)$ – j-ojo ilgio ciklų skaičius skaidinyje (5.2). Funkciją h vadinsime *adytiviąja*, jei ją galima išreikšti šiuo būdu:

$$h(\sigma) = \sum_{j \le n} h_j(\kappa_j(\sigma)),$$

čia $h_j\colon \mathbb{N}_0 \to \mathbb{R}$ – funkcijos, tenkinančios $h_j(0) = 0$, $j \le n$. Be to, funkciją h vadinsime *stipriai adityviąja*, jei $h_j(k) = a_j \mathbf{1}\{k \ge 1\}$, $a_j \in \mathbb{R}$, ir *visiškai adytiviąja*, jei $h_j(k) = h_j(1)k$, $k \in \mathbb{N}$.

Toliau vartosime šiuos atsitiktinių keitinių teorijos žymenis. Tegu ν_n yra tolygusis aibėje $\mathbb{S}_n$ apibrėžtas tikimybinis matas, $\mathbf{E}_n$ ir $\mathbf{V}_n$ atitinkamai žymi vidurkį ir dispersiją ν_n atžvilgiu. Įvedus pastarąjį, struktūrinio vektoriaus koordinatės $\kappa_j := \kappa_j(\sigma)$, $j \le n$, tampa atsitiktiniais dydžiais. Apibrėžkime atvaizdį $\ell(\bar{s}) := 1s_1 + \cdots + ns_n$, kai $\bar{s} = (s_1, \dots, s_n) \in \mathbb{N}_0^n$. Tuomet $\ell\big(\bar{\kappa}(\sigma)\big) = n$ kiekvienam $\sigma \in \mathbb{S}_n$.

Paminėsime svarbiausius istorinius atsitiktinių keitinių teorijos ir šio darbo temai artimus rezultatus. Tegu ξ_j, $j \le n$, yra nepriklausomi a.d.

pasiskirstę pagal Puasono dėsnį su parametrais $1/j$ ir $\bar{\xi} := (\xi_1, \ldots, \xi_n)$. Tada turime

$$\nu_n(\bar{\kappa} = \bar{s}) = \mathbf{1}\{\ell(\bar{s}) = n\} \prod_{j \leq n} \frac{1}{j^{s_j} s_j!} = P\big(\bar{\xi} = \bar{s} | \ell(\bar{\xi}) = n\big). \tag{5.3}$$

Šią, *sąlyginiu sąryšiu* vadinamą, lygybę pirmą kartą atskleidė V.L. Goncharov [11, 12]. Pasinaudoję (5.3), galime įvertinti įvairias adityviąsias atsitiktinių keitinių statistikas per nepriklausomų a.d. funkcionalus. Autorius taip pat tyrė visiškai adityviąją funkciją $\omega(\sigma)$, kurios vidurkiui ir dispersijai gavo asimptotinius įverčius ir jais remiantis įrodė centrinę ribinę teoremą (CRT). Šiuos rezultatus galima laikyti atsitiktinių keitinių teorijos pradžia.

Kitas svarbus rezultatas priklauso P. Erdös ir P. Turán [9]. Atsitiktinai ištraukto keitinio σ eilės grupėje $\mathbb{S}_n$ logaritmą jie aproksimavo visiškai adityviąja funkcija

$$\sum_{j \leq n} \kappa_j(\sigma) \log j.$$

Atitinkamai normuota ir centruota ši funkcija turi ribinį normalųjį skirstinį, todėl ir atsitiktinio keitinio σ grupinės eilės logaritmas paklūsta tam pačiam dėsniui. Iki šiol geriausią konvergavimo greitį šioje CRT gavo V. Zacharovas [51]. Jo darbe buvo pritaikytas analizinis metodas, anksčiau naudotas bendrų visiškai adityviųjų funkcijų ribinėse teoremose [30].

Dar 1996 m. E. Manstavičius pradėjo tirti adityviųjų funkcijų integralinius pasiskirstymo dėsnius. Buvo keliamas klausimas: kada $\nu_n(h_n(\sigma) - A(n) < x)$ silpnai konverguoja į kažkokį ribinį dėsnį? Čia $h_n(\sigma)$ – visiškai adityviųjų funkcijų seka, $A(n)$ – centruojanti seka, artima $\mathbf{E}_n h_n(\sigma)$. Straipsnyje [35] autorius rado silpnojo didžiųjų skaičių dėsnio galiojimo būtinas ir pakankamas sąlygas. R. Arratia ir S. Tavaré [2] pateikė išsamią tikimybinę atsitiktinių keitinių struktūros vektoriaus analizę. Ji yra moderniosios adityviųjų funkcijų aproksimavimo Puasono a.d. metodikos pagrindas. Šis ir kiti dvidešimto amžiaus rezultatai aprašomi knygoje [1].

Dabar apžvelgsime keletą naujesnių darbų, kuriuose gaunami adityviųjų funkcijų dispersijų įverčiai. J. Klimavičius ir E. Manstavičius [19] gavo tikslų rezultatą visiškai adityviosioms funkcijoms, apibrėžtoms virš atsitiktinių keitinių. Tegul $h(\sigma) = \sum_{j\le n} a_j\kappa_j(\sigma)$ – visiškai adityvioji funkcija. Pažymėkime

$$B_n(\bar a) := \sum_{j\le n}\frac{a_j^2}{j},\ \bar a\in\mathbb{R}^n.$$

Pastebėkime, kad $B_n(\bar a)$ lygi nepriklausomų atsitiktinių dydžių $a_j\xi_j,\ j\le n$, dispersijų sumai. Tuomet

$$\sup\Big\{\mathbf{V}_n h/B_n(\bar a):\ \bar a\ne 0,\, n\ge 2\Big\} = \frac{3}{2}.$$

Įrodymas yra pagrįstas matricos, susijusios su kvadratine forma $\mathbf{V}_n h$, minimalios tikrinės reikšmės paieška. Ją atitinkantys tikriniai vektoriai yra išreiškiami Hahn'o polinomais [37]. Skaičių teorijoje tokioje vietoje buvo naudojami Jacobi'o polinomai [26]. Šį rezultatą apibendrino Ž. Baronėnas, E. Manstavičius ir P. Šapokaitė [4]. Vietoje tolygaus keitinių pasiskirstymo, buvo naudojamas Ewens'o tikimybinis matas $\nu_{n,\theta}(\{\sigma\}) = \theta^{\omega(\sigma)}/(\theta)_n$, čia $(x)_n = x(x-1)\cdots(x-n+1)$ žymi krentantįjį faktorialą. Tegu $\Theta(m) = (\theta)_m/m!,\ m\in\mathbb{N}$. Tuomet visiškai adityviosios funkcijos $h(\sigma)$ dispersija Ewens'o tikimybinio mato atžvilgiu lygi

$$\mathbf{V}_{n,\theta} h = \theta\sum_{j\le n}\frac{a_j^2}{j}\frac{\Theta(n-j)}{\Theta(n)} + \theta^2\sum_{i+j\le n}\frac{a_ia_j}{ij}\frac{\Theta(n-i-j)}{\Theta(n)} - \theta^2\sum_{j\le n}\bigg(\sum_{j\le n}\frac{a_j}{j}\frac{\Theta(n-j)}{\Theta(n)}\bigg)^2.$$

Buvo įrodyta, kad

$$\sup\left\{\frac{\mathbf{V}_{n,\theta} h}{B_{n,\theta}(\bar a)}:\ \bar a\ne 0,\, n\ge 2\right\} = \frac{\theta+2}{\theta+1},$$

čia

$$B_{n,\theta}(\bar a) = \theta\sum_{j\le n}\frac{a_j^2}{j}\frac{\Theta(n-j)}{\Theta(n)}.$$

Parinkę $\theta = 1$ gauname jau anksčiau minėtą rezultatą [19]. Bendroms adityviosioms funkcijoms geriausią dispersijos įvertį, bet be asimptotinės konstantos išraiškos, gavo E. Manstavičius ir V. Stepas [39].

Bendrų adityviųjų funkcijų laipsninius momentus nagrinėjo E. Manstavičius [33]. Atitinkami rezultatai Ewens'o tikimybinio mato atžvilgiu įrodyti darbe [40]. Abiejuose straipsniuose gauti įverčiai priklauso tik nuo momento laipsnio ir yra tolygūs adityviųjų funkcijų atžvilgiu.

Adityviųjų funkcijų teorija yra plėtojama ne vien tik atsitiktiniams keitiniams, bet ir kitoms skaidžiosioms kombinatorinėms struktūroms. Šios disertacijos kontekste mus taip pat domina aritmetinių pusgrupių teorija, formalizuota J. Knopfmacher [21]. Rezultatus, gautus pusgrupiams ar svorinių multiaibių klasėms, laikysime ekvivalenčiais (žr. [20], [10, II-as skyrius]).

Daugybė matematikų prisidėjo prie tikimybinės adityviųjų funkcijų teorijos aritmetiniams pusgrupiams plėtojimo ([15], [14], [13], [38], [43], [22], [20]). Išskirsime W.-B. Zhang įnašą ([52], [53], [54]). Darbe [52] jis tyrė adityviųjų funkcijų skirstinių silpnąjį konvergavimą bei gavo dispersijos įvertį, atitinkantį Turáno–Kubiliaus nelygybę skaičių teorijoje. S. Wehmeier savo disertacijoje [47, 3-ias skyrius] ir darbe [48], apibendrino šį rezultatą stipriai adityviosioms funkcijoms, apibrėžtoms bendresnėje puspgrupių klasėje. Jį cituojame naudodami autoriaus originalius žymėjimus.

Tegu $(G,\cdot)$ yra laisvasis komutatyvus monoidas, apibrėžtas virš netuščios "pirminių aibės" P. Apibrėžkime laipsnio funkciją $\partial : G \to \mathbb{N}_0$, tenkinančią sąlygą $\partial(ab) = \partial(a) + \partial(b)$ kiekvienam $a, b \in G$. Tegu $G_n = \{a \in G : \partial(a) = n\}$ su sąlyga, kad $G(n) = |G_n|$ yra baigtinis ir $f : G \to \mathbb{R}$, toks atvaizdis, kad $f(a) = \sum_{p|a} f(p)$, $a \in G$. Tarkime

$$G(n) = (A + O((\log n)^{-1})q^n,$$

čia $q > 1$ ir $A > 0$. Be to, tarsime, kad galioja Čebyševo sąlyga:

$$P(k) \le C_G \max_{\substack{n\in\mathbb{N} \\ G(n-k)>0}} \frac{G(n)}{kG(n-k)},$$

kiekvienam $k \in \mathbb{N}$ su kokia nors teigiama konstanta C_G, čia $P(k) = |\{p \in P : \partial(p) = k\}|$. Pažymėsime

$$A(n) = \frac{1}{G(n)} \sum_{\partial(p)\le n} f(p)G(n - \partial(p)),$$

$$B^2(n) = \frac{1}{G(n)} \sum_{\partial(p)\le n} f(p)^2 G(n - \partial(p)).$$

Tuomet

$$\frac{1}{G(n)}\sum_{a\in G_n}(f(a)-A(n))^2=O(B^2(n)).$$

Konstanta simbolyje $O(\cdot)$ priklauso tik nuo pusgrupio parametrų. Paprastą įvertinio išraišką, esant bendroms sąlygoms, būtų sunku rasti, todėl disertacijoje mes apsiribojame siauresne pusgrupių šeima, bet gauname efektyvų dispersijos įvertį su tikslia asimptotine konstanta.

Pagrindiniai rezultatai

Stipriai adityviosios funkcijos, apibrėžtos virš atsitiktinių keitinių

Išdėstysime rezultatus, gautus stipriai adityviosioms funkcijoms h, apibrėžtoms virš atsitiktinių keitinių. Naudosime tokią išraišką

$$h(\sigma)=\sum_{j\le n}a_j\varkappa_j,\quad \varkappa_j=\varkappa_j(\sigma):=\mathbf{1}\{\kappa_j(\sigma)\ge 1\},\quad j\le n.$$

Atkreipkime dėmesį, kad visiškai adityviųjų funkcijų skaidinyje vietoje indikatorių būtų a.d. $\kappa_j(\sigma)$, $j\le n$, kurie, nors būdami priklausomi, yra mažiau koreliuoti negu dabar naudojami Bernulio a.d. $\varkappa_j$, įgyjantys reikšmę 1 su tikimybėmis

$$p_j(n):=\nu_n(\kappa_j\ge 1)=\mathbf{E}_n\varkappa_j=\sum_{1\le k\le n/j}\frac{(-1)^{k+1}}{j^k k!}.$$

Dabar turime a.d. poros kovariaciją

$$\begin{aligned}\mathbf{C}ov_n(\varkappa_i,\varkappa_j)&=\nu_n(\varkappa_i=1,\varkappa_j=1)-\nu_n(\varkappa_i=1)\nu_n(\varkappa_j=1)\\&=\sum_{\substack{l,k\ge 1\\ il+jk\le n}}\frac{(-1)^{l+k}}{i^l l! j^k k!}-\sum_{1\le l\le n/i}\frac{(-1)^{l+1}}{i^l l!}\cdot\sum_{1\le k\le n/j}\frac{(-1)^{k+1}}{j^k \kappa!}\\&=: p_{i,j}(n)-p_i(n)p_j(n),\quad i\ne j.\end{aligned}$$

Išraiška yra žymiai sudėtingesnė nei visiškai adityviųjų funkcijų atveju, kai

$$\mathbf{C}ov_n(\kappa_i,\kappa_j)=\frac{\mathbf{1}\{i=j\}}{j}-\frac{\mathbf{1}\{i+j>n\}}{ij}.$$

Pasinaudoję $\mathbf{C}ov_n(\varkappa_i, \varkappa_j)$ formule, gauname stipriai adityviosios funkcijos dispersiją

$$\begin{aligned}
\mathbf{V}_n h =: V_n(\bar{a}) &= \frac{1}{n!}\sum_{\sigma\in\mathbb{S}_n} h^2(\sigma) - \Big(\sum_{j\le n} a_j p_j(n)\Big)^2 \\
&= \sum_{j=1}^{n} a_j^2 p_j(n) - \sum_{\substack{i,j\le n\\ i+j>n}} a_i a_j p_i(n) p_j(n) \\
&\quad - \sum_{\substack{i+j\le n\\ i\ne j}} a_i a_j [p_i(n)p_j(n) - p_{i,j}(n)] - \sum_{j\le n/2} a_j^2 p_j^2(n) \\
&=: D_n(\bar{a}) - \Delta_{n1}(\bar{a}) - \Delta_{n2}(\bar{a}) - \Delta_{n3}(\bar{a}). \qquad (5.4)
\end{aligned}$$

Dabar pagrindinis tikslas yra gauti dispersijos $\mathbf{V}_n h$ viršutinį įvertį, priklausantį tik nuo pirmojo dėmens $D_n(\bar{a})$, kuris artimas $\sum_{j\le n} \mathbf{V}_n(a_j\varkappa_j)$. Iš tiesų,

$$B_n(\bar{a})/4 \le D_n(\bar{a})/2 \le \sum_{j\le n} \mathbf{V}_n(a_j\varkappa_j) \le D_n(\bar{a}) \le B_n(\bar{a}).$$

Lengva įrodyti, jog praleidę paskutinę neteigiamą kvadratinę formą, įvedame tik nežymią paklaidą

$$D_n(\bar{a}) - \Delta_{n3}(\bar{a}) - \sum_{j\le n} \mathbf{V}_n(a_j\varkappa_j) = \sum_{n/2<j\le n} a_j^2 p_j^2(n) \le \frac{2}{n} D_n(\bar{a}).$$

Likusios kvadratinės formos įvertinamos nepagerinamos asimptotinės konstantos tikslumu. Be to, gaunamas ir apatinis dispersijos įvertis.

Teorema 5.1. *Tegu $\bar{a}\in\mathbb{R}^n$ yra bet koks ir $B_n(\bar{a})$ toks kaip praeitame skyriuje. Jei*

$$h := h(\sigma) := \sum_{j\le n} a_j\varkappa_j, \quad \sigma\in\mathbb{S}_n,$$

tuomet

$$\mathbf{V}_n h \le \Big(\frac{3}{2} + R(n)\Big) D_n(\bar{a}) \le \Big(\frac{3}{2} + R(n)\Big) B_n(\bar{a}), \qquad (5.5)$$

čia $R(n) = O(n^{-1/3})$ su absoliučia neišreikštine konstanta.

Taip pat, jei $\bar{a}$ yra lygties $\sum_{j\le n} jx_j p_j(n) = 0$ sprendinys, tai

$$\mathbf{V}_n h \ge \frac{2}{3} B_n(\bar{a})\big(1 + R(n)\big). \qquad (5.6)$$

Remiantis (5.4), teoremos 5.1 įrodymas susideda iš kvadratinių formų analizės. Pirma kvadratinė forma Δ_{n1} lengvai įvertinama pritaikius straipsnio [19] nelygybes, o Δ_{n2} nagrinėjama tiriant atitinkamos matricos kvadrato pėdsaką. Konstantos 3/2 ir 2/3 yra nepagerinamos, t.y., egzistuoja vektoriai $\bar{a}_2$ ir $\bar{a}_3$, su kuriais nelygybės (5.5) ir (5.6) tampa lygybėmis. Naudodami sudėtingesnį metodą, asimpotinį greitį $O(n^{-1/3})$ pageriname iki $O(n^{-1})$.

Teorema 5.2. *Tegu galioja teoremos 5.1 žymenys. Tuomet lygybės (5.5) ir (5.6) yra teisingos su liekamuoju nariu $R(n) = O(n^{-1})$.*

Pastarosios teoremos įrodymas yra pagrįstas laipsninio metodo idėja didžiausiai matricos tikrinei reikšmei rasti. Jau po antrosios algoritmo iteracijos pasiekiame norimą asimptotinį greitį. Be to, dar tenka pasinaudoti kvadratinės formos Δ_{n2} matricos pėdsako įverčiu, gautu įrodant teoremą 5.1.

Toliau pateiksime alternatyvią teoremos 5.2 formuluotę. Įvedę keitinį $a_j = \sqrt{j}b_j$, $j \le n$, gauname

$$||\bar{b}\Gamma_n||^2 = (\bar{b}\Gamma_n)(\Gamma_n'\bar{b}') \le \big(3/2 + O(n^{-1})\big)n!||\bar{b}||^2, \tag{5.7}$$

visiems $\bar{b} \in \mathbb{R}^n$. Matricos transponavimą žymėsime $'$. Tegu $\Gamma_n = ((\gamma_{j\sigma}))$ yra $n \times n!$ matrica, kurios elementai

$$\gamma_{j\sigma} = \sqrt{j}\big(\varkappa_j(\sigma) - 1/j\big), \quad j \le n, \ \sigma \in \mathbb{S}_n.$$

Galima pastebėti, kad (5.7) tėra efektyvus matricos $\Gamma_n\Gamma_n'$ maksimalios tikrinės reikšmės įvertis, sutampantis su atitinkamu $n! \times n!$ matricos $\Gamma_n'\Gamma_n$ įverčiu (žr. lemą 3.3, 30 psl. [8]). Tad gauname dualiąją nelygybės (5.7) formą.

Išvada 5.3. *Visiems vektoriams $\bar{d} \in \mathbb{R}^{n!}$ su bendrine koordinate d_σ, $\sigma \in \mathbb{S}_n$, teisingas šis įvertis*

$$\sum_{j\le n} j\bigg(\sum_{\sigma\in\mathbb{S}_n} d_\sigma\big(\varkappa_j(\sigma) - 1/j\big)\bigg)^2 \le \big(3/2 + O(n^{-1})\big)n!||\bar{d}||^2.$$

Šis dualumas, išpopuliarintas P.D.T.A. Elliott [8], dar iki šiol neturi atitikmens atsitiktinių keitinių teorijoje.

Visiškai adityviosios funkcijos, apibrėžtos virš svorinių multiaibių

Tegu $(\mathcal{P}, ||\cdot||)$ yra pirminė svorinių objektų aibė, tenkinanti sąlygą

$$\pi(j) := \big|\{p \in \mathcal{P} : \; ||p|| = j\}\big| < \infty$$

kiekvienam $j = 1, 2, \ldots$. Nagrinėkime aibę $\mathcal{G}$ multiaibių, sudarytų iš $p \in \mathcal{P}$ elementų su galimais pasikartojimais, ir joje natūraliai išplėskime svorio funkciją $||\cdot||$. Jei $\varrho \in \mathcal{G}$ ir $\varrho = \{p_1, \ldots, p_r\}$, tai $||\varrho|| = ||p_1|| + \cdots + ||p_r||$. Tuščiai multiaibei $\emptyset$ priskirkime svorį 0. Tuomet n-ojo svorio multiaibių skaičių žymėsime

$$m(n) := |\mathcal{G}_n| := \big|\{\varrho \in \mathcal{G} : \; ||\varrho|| = n\}\big| = \sum_{\ell(\bar{k})=n} \prod_{j=1}^{n} \binom{\pi(j) + k_j - 1}{k_j},$$

čia $\ell(\bar{k}) = 1k_1 + \cdots + nk_n$, $\bar{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ ir $n \in \mathbb{N}_0$.

Toliau nagrinėsime klasę multiaibių, tenkinančių sąlygą $m(n) = q^n$, čia $q \geq 2$ yra natūralusis skaičius. Dažnai aritmetinių pusgrupių ar svorinių multiaibių analizės pagrindas yra silpnesnė aksioma $\mathcal{A}^{\#}$ (žr. [22]), bet stipresnė negu S. Wehmeier darbe [47] įvestos sąlygos. Aksioma $\mathcal{A}^{\#}$ formuluojama taip:

$$m(n) = Aq^n + R_n, \quad A \in \mathbb{R}_+, \quad R_n = O(q^{cn}), \quad 0 \leq c < 1, \quad \text{as } n \to \infty.$$

Savo darbe pasirinkome griežtesnę aksiomą siekdami supaprastinti vėliau apibrėžiamų adityviųjų funkcijų dispersijos išraiškas.

Jei q yra pirminio skaičiaus laipsnis, tuomet $\mathcal{G}$ galėtų būti interpretuojamas kaip moninių polinomų žiedas $\mathbb{F}_q^*[t]$ virš baigtinio kūno $\mathbb{F}_q$. Tokiu atveju $\mathcal{P}$ būtų neredukuojamų polinomų aibė, o svorio funkcija – polinomo laipsnis. Bet kokiam natūraliajam $q \geq 2$, aibė $\mathcal{P}$ gali būti sudaroma iš specifinių kombinatorinių struktūrų – vėrinių virš q raidžių abėcėlės. Tokios vėrinių multiaibės tenkina sąlygą $m(n) = q^n$ (daugiau apie vėrinius galima rasti [1] knygoje, žr. 2.12 skyriaus 43 psl.). Joms, kaip ir polinomams, teisingos šios lygybės:

$$\pi(n) = \frac{1}{n} \sum_{d|n} q^{n/d} \mu(d), \qquad q^n = \sum_{d|n} d\pi(d),$$

čia d perbėga visus natūraliuosius n daliklius, $\mu(d)$ – Möbius'o funkcija. Šie sąryšiai yra ekvivalentūs formaliai begalinių eilučių lygybei

$$\sum_{n=0}^{\infty} q^n x^n = \frac{1}{1-qx} = \prod_{j=1}^{\infty} (1-x^j)^{-\pi(j)}.$$

Atsitiktinai su vienoda tikimybe imkime multiaibę $a \in \mathcal{G}_n$, t.y. apibrėžkime tikimybę $\nu_n(\{\varrho\}) = q^{-n}$, $n \in \mathbb{N}$ ir $\nu_0(\{\emptyset\}) = 1$ (čia tikimybinis matas ν_n skiriasi nuo tikimybinio mato apibrėžto virš keitinių). Jei $k_j(\varrho) \geq 0$ yra j-ojo svorio pirminių svorinių objektų skaičius multiaibėje $\varrho \in \mathcal{G}_n$, tuomet $\bar{k}(\varrho) = \big(k_1(\varrho), \ldots, k_n(\varrho)\big)$ žymės svorinės multiaibės $\varrho \in \mathcal{G}_n$ struktūrinį vektorių, tenkinantį $\ell(\bar{k}(\varrho)) = n$. Jo skirstinys turi pavidalą

$$\nu_n\big(\bar{k}(\varrho) = \bar{s}\big) = \mathbf{1}\{\ell(\bar{s}) = n\} q^{-n} \prod_{j=1}^{n} \binom{\pi(j) + s_j - 1}{s_j},$$

čia $\bar{s} = (s_1, \ldots, s_n) \in \mathbb{N}_0^n$ ir $\mathbf{1}\{\cdot\}$ – indikatoriaus funkcija.

Šiame darbe nagrinėjame *visiškai adityviųjų funkcijų* (vartojama ir *tiesinių statistikų*), apibrėžtų virš svorinių multiaibių, dispersiją tikimybinio mato ν_n atžvilgiu

$$h(\bar{a}) := h(\bar{a}, \varrho) = a_1 k_1(\varrho) + \cdots + a_n k_n(\varrho), \quad \bar{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n.$$

Komponenčių skaičius multiaibėje $\varrho \in \mathcal{G}_n$, kuris lygus $k_1(\varrho) + \cdots + k_n(\varrho)$ yra paprasčiausias šių funkcijų pavyzdys. Daugiau jų galima rasti [1]. Pagrindinės kliūtys vertinant šių funkcijų dispersiją kyla iš tiesinę statistiką sudarančių atsitiktinių dydžių priklausomumo.

Toliau darbe vidurkį ir dispersiją mato ν_n atžvilgiu žymėsime $\mathbf{E}_n$ ir $\mathbf{V}_n$; kai tikimybinė erdvė $(\Omega, \mathcal{F}, P)$ nenurodyta naudosime žymenis $\mathbf{E}$ ir $\mathbf{V}$. Visi sumavimo indeksai i, j, l, k, m, m_1 ir m_2 yra natūralieji skaičiai.

Teorema 5.4. *Jei $\bar{a} \in \mathbb{R}^n$ ir $n \in \mathbb{N}_0$, tai*

$$\mathbf{V}_n h(\bar{a}) = \sum_{1 \leq jk \leq n} a_j^2 \pi(j) k q^{-jk} - \sum_{\substack{il+jk>n \\ il \leq n, jk \leq n}} a_i a_j \pi(i) \pi(j) q^{-il-jk}. \tag{5.8}$$

Žinoma [1], jog fiksuotam j a.d. $k_j(\varrho)$ konverguoja pagal skirstinį į a.d. γ_j, kuris pasiskirstęs pagal neigiamą binominį dėsnį $NB(\pi(j), q^{-j})$.

Tegu $\{\gamma_1, \gamma_2, \dots\}$ yra tarpusavyje nepriklausomi. Apibrėžkime jų tiesinę kombinaciją $Y_n = a_1\gamma_1 + \cdots + a_n\gamma_n$. Darbe įrodysime, jog pirmoji dešiniosios pusės suma (5.8) yra gerai aproksimuojama $\mathbf{V}Y_n$, todėl vertindami $\mathbf{V}_n h(\bar{a})$, naudosime šias kvadratines formas:

$$\mathbf{B}_n(\bar{a}) := \sum_{1\le jk\le n} a_j^2 \pi(j) k q^{-jk}, \qquad \mathbf{R}_n(\bar{a}) := \sum_{m\le n} m q^{-2m} \Big(\sum_{j|m} a_j \pi(j)\Big)^2.$$

Teorema 5.5. *Jei $n \ge 2$, tuomet*

$$\mathbf{V}_n h(\bar{a}) \le \mathbf{B}_n(\bar{a}) + \frac{1}{2}\mathbf{R}_n(\bar{a}).$$

Be to, nelygybė tampa lygybe, kai

$$a_j = a_j^* := \frac{3}{\pi(j)} \sum_{d|j} d q^d \mu\Big(\frac{j}{d}\Big) - 2n - 1, \quad 1 \le j \le n.$$

Išvada 5.6. *Jei $n \ge 2$ ir $\bar{a} \ne \bar{0}$, tai*

$$\mathbf{V}_n h(\bar{a}) < \frac{3}{2}\mathbf{B}_n(\bar{a}) < \Big(\frac{3}{2} - \frac{q-1}{q} n q^{-n}\Big)\mathbf{V}Y_n.$$

Šios nelygybės yra trivialios funkcijoms, proporcingoms $h(\bar{j}, \varrho) = n$, čia $\bar{j} = (1, 2, \dots, n)$, kai $\varrho \in \mathcal{G}_n$, nes tada $\mathbf{V}_n h(\bar{j}) = 0$. Norėdami to išvengti naudosime $\bar{a}$ postūmį. Pastebėkime, jog abi kvadratinės formos $\mathbf{B}_n(\bar{a} - t\bar{j})$ ir $\mathbf{R}_n(\bar{a} - t\bar{j})$ įgyja mažiausią reikšmę pagal $t \in \mathbb{R}$, kai

$$t = t_{\bar{a}} := \frac{2}{(n+1)n} \sum_{m\le n} m q^{-m} \sum_{j|m} a_j \pi(j).$$

Teorema 5.7. *Jei $n \ge 3$, tai*

$$\mathbf{B}_n(\bar{a} - t_{\bar{a}}\bar{j}) - \frac{1}{3}\mathbf{R}_n(\bar{a} - t_{\bar{a}}\bar{j})) \le \mathbf{V}_n h(\bar{a}) \le \mathbf{B}_n(\bar{a} - t_{\bar{a}}\bar{j}) + \frac{1}{2}\mathbf{R}_n(\bar{a} - t_{\bar{a}}\bar{j}). \quad (5.9)$$

Abi (5.9) nelygybės yra optimalios, kai dispersijos įvertyje naudojama kvadratinė forma $\mathbf{R}_n$.

Išvada 5.8. *Jei $n \ge 3$ ir $\bar{a} \notin \mathbb{R}(\bar{j})$, tai*

$$\frac{2}{3}\mathbf{B}_n(\bar{a} - t_{\bar{a}}\bar{j}) < \mathbf{V}_n h(\bar{a}) < \frac{3}{2}\mathbf{B}_n(\bar{a} - t_{\bar{a}}\bar{j}),$$

čia $\mathbb{R}(\bar{j})$ – tiesinis vektoriaus $\bar{j}$ apvalkas.

Be šių absoliučių dispersijos įverčių, taip pat gaunamos santykio $\frac{\mathbf{V}_n h(\bar{a})}{\mathbf{B}_n(\bar{a})}$ supremumo ir infimumo asimptotikos.

Teorema 5.9. *Egzistuoja toks $C > 0$, kad*

$$\frac{3}{2} - \frac{C}{n^2} \leq \sup_{\bar{a} \in \mathbb{R}^n \setminus \{\bar{0}\}} \frac{\mathbf{V}_n h(\bar{a})}{\mathbf{B}_n(\bar{a})} \leq \frac{3}{2},$$

kai $n \geq 2$.

Teorema 5.10. *Egzistuoja toks $C > 0$, kad*

$$\frac{2}{3} \leq \inf_{\bar{a} \in \mathbb{R}^n \setminus \mathbb{R}(\bar{j})} \frac{\mathbf{V}_n h(\bar{a})}{\mathbf{B}_n(\bar{a})} \leq \frac{2}{3} + \frac{C}{n^2},$$

kai $n \geq 3$.

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