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VILNIUS UNIVERSITY

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# Sums of Heavy-Tailed Random Variables and Haezendonck-Goovaerts Risk Measure

**DOCTORAL DISSERTATION**

Natural Sciences,
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VILNIAUS UNIVERSITETAS

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# Sunkiauodegių atsitiktinių dydžių sumos ir Haezendonck - Goovaerts rizikos matas

**DAKTARO DISERTACIJA**

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## NOTATION

| | |
|---|---|
| $\mathbb{N}$ | The set of natural numbers, i.e. $1, 2, \ldots$; |
| $\mathbb{N}_0$ | The set of natural numbers including 0, i.e. $0, 1, 2, \ldots$; |
| $\mathbb{R}$ | The set of real numbers; |
| $x^+$ | The positive part of the real number $x$, i.e. $x^+ := \max\{0, x\}$; |
| $x^-$ | The negative part of the real number $x$, i.e. $x^- := -\min\{0, x\}$; |
| $\lfloor x \rfloor$ | The integer part of the real number $x$; |
| $\{x\}$ | The fractional part of the real number $x$, i.e. $\{x\} := x - \lfloor x \rfloor$; |
| $\mathbb{1}_A(\omega)$ | The indicator function of $A$: $\mathbb{1}_A(\omega) = 1$ if $\omega \in A$, $\mathbb{1}_A(\omega) = 0$ otherwise; |
| $S_n^{\xi}$ | The sum of random variables $\xi_1, \ldots, \xi_n$, $n \in \mathbb{N}$, i.e. $S_n^{\xi} := \xi_1 + \ldots + \xi_n$; |
| $S_n^{\theta\xi}$ | The weighted sum of random variables $\xi_1, \ldots, \xi_n$ and $\theta_1, \ldots, \theta_n$, $n \in \mathbb{N}$, i.e. $S_n^{\theta\xi} := \theta_1\xi_1 + \ldots + \theta_n\xi_n$; |
| $\mathbb{P}(A)$ | The probability of the event $A$; |
| $\mathbb{E}(X)$ | The expectation of the random variable $X$; |
| $F_X(x)$ | The distribution function of the random variable $X$, i.e. $F_X(x) := \mathbb{P}(X \leqslant x)$, when the context is clear, the notation is shortened to $F(x)$; |
| $\overline{F}_X(x)$ | The tail function of the random variable $X$, i.e. $\overline{F}_X(x) := 1 - F_X(x) = \mathbb{P}(X > x)$, when the context is clear, the notation is shortened to $\overline{F}(x)$; |
| $F_X * F_Y$ | The convolution of distribution functions $F_X$ and $F_Y$, i.e. if $X$ and $Y$ are independent, then $F_X * F_Y(x) := \mathbb{P}(X + Y \leqslant x)$; |
| $\overline{F_X * F_Y}$ | The tail function of $F_X$ and $F_Y$ convolution, i.e. if $X$ and $Y$ are independent, then $\overline{F_X * F_Y} := \mathbb{P}(X + Y > x)$; |
| $F^{*n}(x)$ | The $n$-fold convolution of the distribution function $F$; |
| $\overline{F_X^{*n}}(x)$ | The tail function of the $n$-fold convolution; |
| $F_X \otimes F_Y$ | The product convolution of distribution function $F_X$ and $F_Y$, i.e. if $X$ and $Y$ are independent, then $F_X \otimes F_Y(x) := \mathbb{P}(XY \leqslant x)$; |
| $\overline{F_X \otimes F_Y}$ | The tail function of $F_X$ and $F_Y$ product convolution, i.e. if $X$, $Y$ are independent, then $\overline{F_X \otimes F_Y}(x) := \mathbb{P}(XY > x)$; |

| | |
|---|---|
| $\mathrm{supp}F$ | The support of the distribution function $F$, i.e. $\mathrm{supp}F := \{x : F(x+\delta) - F(x-\delta) > 0 \quad \text{for all } \delta > 0\}$; |
| $\mathscr{H}$ | The class of heavy-tailed distributions; |
| $\mathscr{L}$ | The class of long-tailed distributions; |
| $\mathscr{S}$ | The class of subexponential distributions; |
| $\mathscr{D}$ | The class of dominatedly varying distributions; |
| $\mathscr{C}$ | The class of consistently varying distributions; |
| $\mathscr{ERV}_{\{\alpha,\beta\}}$ | The class of extended regularly varying distributions with indices $0 \leqslant \alpha \leqslant \beta < \infty$; |
| $\mathscr{R}_\alpha$ | The class of regularly varying distributions with index $\alpha \geqslant 0$; |
| $\mathcal{PD}$ | The class of distributions with a positively decreasing tail; |
| $\underset{x\to\infty}{\sim}$ | It is written $f(x) \underset{x\to\infty}{\sim} g(x)$ if $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$, where $f, g$ are real functions and positive for sufficiently large $x$; |
| $\underset{x\to\infty}{\lesssim}$ | It is written $f(x) \underset{x\to\infty}{\lesssim} g(x)$ if $\limsup\limits_{x\to\infty} \dfrac{f(x)}{g(x)} \leqslant 1$, where $f$, $g$ are real functions and positive for sufficiently large $x$; |
| $o(\cdot)$ | It is written $f(x) = o(g(x))$ if $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$, where $f, g$ are real functions and positive for sufficiently large $x$; |
| $O(\cdot)$ | It is written $f(x) = O(g(x))$ if $\limsup\limits_{x\to\infty} \dfrac{f(x)}{g(x)} < \infty$, where $f, g$ are real functions and positive for sufficiently large $x$; |
| $\asymp$ | It is written $f(x) \asymp g(x)$ if $f(x) = O(g(x))$ and $g(x) = O(f(x))$, where $f, g$ are real functions and positive for sufficiently large $x$. |

CONTENTS

# INTRODUCTION

## Research Object

The mathematical research presented in this doctoral dissertation is primarily concerned with the asymptotic analysis of random sums composed of heavy-tailed random variables.

More specifically, the focus is placed on several important classes of random sums. First, sums with a fixed number of summands are considered, both in the unweighted form

$$S_n^{\xi} = \xi_1 + \ldots + \xi_n$$

and in the randomly weighted form

$$S_n^{\theta\xi} = \theta_1\xi_1 + \ldots + \theta_n\xi_n.$$

Second, randomly stopped sums

$$S_\nu := \begin{cases} 0 & \text{if } \nu = 0, \\ \xi_1 + \ldots + \xi_\nu & \text{if } \nu \geqslant 1, \end{cases}$$

are examined, where the number of summands $\nu$ is itself a random variable.

A unifying feature of the sums studied in this dissertation is the assumption that the summands or the stopping moment exhibit heavy-tailed behaviour. Heavy-tailed distributions are characterised by a slow decay of tail probabilities and play a crucial role in modelling extreme events. In contrast to light-tailed distributions, heavy-tailed models allow for the possibility that large individual losses dominate aggregate outcomes - a phenomenon frequently observed in insurance and financial data. As a consequence, classical tools based on exponential moments or moment generating functions are often inapplicable, necessitating alternative asymptotic techniques.

Within this framework, the dissertation further investigates risk measurement for aggregate losses. Interpreting the summands as individual random losses, attention is devoted to the Haezendonck–Goovaerts risk measure, a coherent risk measure defined on Orlicz spaces. This measure provides a flexible framework for modelling risk aversion and tail sensitivity, allowing the influence of extreme losses to be controlled through the choice of a Young function. The dissertation analyses the asymptotic behaviour of the Haezendonck–Goovaerts risk measure applied to sums of heavy-tailed risks and explores how this behaviour is determined by the dominant components of the sum.

## Aims and Problems

Three principal objectives are formulated in this dissertation and are briefly outlined in this section.

First, for random and randomly weighted sums with a fixed number of summands, the asymptotic behaviour of the generalized left truncated moments $\mathbb{E}(\varphi(S_n^{\xi})\mathbb{1}_{\{S_n^{\xi}>x\}})$ and $\mathbb{E}(\varphi(S_n^{\theta\xi})\mathbb{1}_{\{S_n^{\theta\xi}>x\}})$, as $x \to \infty$, is investigated. Here $\varphi : \mathbb{R} \to \mathbb{R}_+ = [0, \infty)$ denotes a transformation function satisfying certain conditions. One of the main aims of the dissertation is to derive asymptotic lower and upper bounds for these generalized truncated moments in terms of sums of the corresponding moments of the individual summands, $\mathbb{E}(\varphi(\xi_k)\mathbb{1}_{\{\xi_k>x\}})$ or $\mathbb{E}(\varphi(\theta_k\xi_k)\mathbb{1}_{\{\theta_k\xi_k>x\}})$, up to explicit multiplicative correction constants. Problems of this type have been previously investigated in the literature. In particular, Dirma et al. [33] established asymptotic lower and upper bounds in the case where the transformation function $\varphi$ is a degree function. The aim of this dissertation is to further generalize these findings by permitting a broader class of transformations.

Second, the research is extended to the study of randomly stopped sums $S_\nu$. More specifically, the inverse-type problems are considered, in which regularity conditions are imposed on the distribution of $S_\nu$. The dissertation addresses the question of how the membership of the distribution of $S_\nu$ in certain subclasses of heavy-tailed distributions determines the tail behaviour of the counting random variable $\nu$. Faÿ et al. [39] considered the case where the randomly stopped sum $S_\nu$ belongs to the class of regularly varying distributions $\mathscr{R}_\alpha$ with index $\alpha > 0$, and established conditions under which the counting random variable $\nu$ belongs to the same class. The aim of this dissertation is to extend these results by showing how the regularity of the randomly stopped sum $S_\nu$ - specifically, its membership in broader classes such as extended regular variation $\mathscr{ERV}$ or dominatedly varying distributions $\mathscr{D}$ - determines the regularity of the counting random variable $\nu$ in the dominatedly varying distributions class $\mathscr{D}$.

Finally, the asymptotic behaviour of the Haezendonck–Goovaerts risk measure applied to sums composed of heavy-tailed risks is examined. The dissertation aims to identify conditions under which the asymptotics of the Haezendonck–Goovaerts risk measure are determined by the dominant summands and to derive explicit asymptotic formulas that avoid the need for computationally complex convolution operations. Similar problem has been previously investigated in the literature. In particular, Tang and Yang [105] derived asymptotic expression for the Haezendonck-Goovaerts risk measure in the case where the Young function is a power function $\varphi(t) = t^{\varkappa}$ with $\varkappa \geqslant 1$, and the underlying random risk belongs to the class of regularly varying distributions $\mathscr{R}_\alpha$ with index $\alpha > \varkappa$. The aim of this dissertation is to extend these results by considering sums of random risks for which some or all components belong to a broader subclass of heavy-tailed distributions, namely the class of consistently varying distributions $\mathscr{C}$. In addition, the analysis is extended to the case of randomly weighted sums.

## Relevance and Practical Motivation

Although this dissertation is primarily theoretical in nature, the author would like to emphasize that the relevance of random sums extends well beyond purely theoretical considerations. These sums arise naturally in a wide range of applications in insurance, actuarial science, and financial risk management.

Sums with a fixed number of summands frequently appear in practical problems within probabilistic modelling in insurance and actuarial contexts. One important example is provided by discrete time risk models. In such models, the primary random variables $\xi_k$ may represent the net losses of an insurance company over time periods $(k-1, k]$, defined as the total claim amount minus total premium income during period $(k-1, k]$. The corresponding random weights $\theta_k$ can be interpreted as stochastic discount factors converting future losses at time $k$ to the their present value at time 0. Under this interpretation, the randomly weighted sum $S_n^{\theta\xi}$ represents the present value of the total discounted net loss of the company over the finite time horizon $(0, n]$.

Another important insurance related application arises in the individual risk model. In this framework, an insurance company is assumed to hold a portfolio consisting of $n$ individual policies. The sum $S_n^{\theta\xi}$ is interpreted as the total claim amount generated by the entire portfolio, while each term $\theta_k\xi_k$, $k = 1, \ldots, n$, corresponds to the claim amount associated with the $k$-th policy. Since it is a possible that no claim is incurred for a given policy, the random weight $\theta_k$ is often modelled as an indicator (Bernoulli) random variable describing the occurrence of the $k$-th claim, with $\theta_k = 1$ if the claim has occurred and $\theta_k = 0$ otherwise. The random variable $\xi_k$ corresponds to the claim size of the $k$-th policy given that the claim occurred.

Randomly weighted sums also play a central role in financial risk management, especially in portfolio modelling and capital allocation. A financial portfolio may be viewed as a collection of $n$ distinct risk components, such as asset classes or business lines. In this setting, the random variable $\xi_k$ represents the loss associated with the $k$-th component. Depending on the modelling perspective, $\theta_k$ may represent a stochastic discount factor applied to future losses or a random portfolio weight reflecting the exposure of the portfolio to the $k$-th component. Consequently, the sum $S_n^{\theta\xi}$ may be interpreted either as the present value of the total portfolio loss or as the aggregate weighted loss of the portfolio. In both cases, heavy-tailed behaviour of individual components may lead to situations in which a small number of dominant risks largely determine the overall risk profile.

In addition to sums with a fixed number of summands, randomly stopped sums arise in a variety of applied settings. Such sums appear whenever the number of contributing components is itself random and determined by an external stochastic mechanism. In insurance and finance, this may occur when the duration of a contract, the number of claims, or the lifetime of a portfolio is random. In survival analysis and reliability theory, randomly stopped sums are used to model cumulative quantities accumulated up to a random failure time or termination event.

Finally, the practical relevance of this dissertation is closely linked to risk measurement. Interpreting the summands as individual random losses, the Haezendonck–Goovaerts risk measure provides a flexible and theoretically sound framework for quantifying aggregate risk in the presence of heavy-tailed distributions. Its formulation within the framework of Orlicz space enables control over sensitivity to extreme losses through the choice of an appropriate Young function. However, the practical application of this risk measure is often hindered by the complexity of its computation. The asymptotic results developed in this dissertation demonstrate that, under suitable conditions, the behaviour of the Haezendonck–Goovaerts risk measure for aggregate losses is largely determined by dominant components of the sum. This insight allows for substantial simplifications in risk evaluation and provides a theoretically justified foundation for approximating complex risk measures in applied settings.

## Structure of the Dissertation

The remainder of this dissertation is organized as follows.

Chapter 1 introduces the necessary background material and notation used throughout the dissertation. It provides an overview of heavy-tailed distributions and their subclasses; reviews dependence structures commonly encountered in conjunction with heavy-tailed distributions; recalls key concepts of risk measures and presents the definition of the Haezendonck–Goovaerts risk measure. The definitions collected in this chapter form the foundation for the subsequent analysis.

Chapter 2 focuses on sums with a fixed number of summands. The asymptotic behaviour of generalized left truncated moments for both unweighted and randomly weighted sums is investigated under various assumptions on the tail behaviour of the summands and on the transformation function. The main results of this chapter establish asymptotic lower and upper bounds for generalized truncated moments of aggregate sums in terms of the corresponding moments of individual summands. These results have also been published in [32].

Chapter 3 is devoted to the study of randomly stopped sums. This chapter addresses inverse-type problems concerning the relationship between the tail behaviour of the randomly stopped sum and the counting random variable. In particular, conditions are identified under which membership of the randomly stopped sum to a certain heavy-tailed subclass implies specific tail properties of the stopping moment. These results are based on the publication [91].

Chapter 4 examines the asymptotic behaviour of the Haezendonck–Goovaerts risk measure applied to sums of heavy-tailed risks. It is shown that, under suitable conditions, the asymptotics of the risk measure are determined by the dominant components of the sum, allowing substantial simplifications in its evaluation. The chapter also extends these results to randomly weighted sums. The results presented in this chapter have appeared in [92].

Chapters 2, 3 and 4 share a similar structure. Each chapter begins with a brief review of the relevant literature, followed by the presentation of the main results and their proofs, and concludes with at least one illustrative example demonstrating the applicability of the obtained results. All of the examples provided in these chapters were generated using the Python programming language.

In addition to the main chapters, the dissertation contains a brief summary of the main results and the list of publications associated with the dissertation.

This dissertation is written in English with a summary in Lithuanian provided at the end.

The results presented in this dissertation were obtained by the author, including joint contributions with co-authors of the published articles listed in Chapter DISSEMINATION OF RESULTS.

Finally, the author's curriculum vitae is provided at the end of the dissertation.

# 1 DEFINITIONS AND PRELIMINARIES

Before presenting the related and main results, the notational conventions adopted throughout this dissertation are established, the key definitions and fundamental properties of heavy-tailed distributions are recalled, the dependence structure that is used in the subsequent analysis is described, and the Haezendonck - Goovaerts risk measure is introduced. Additional auxiliary definitions are also provided where necessary.

## 1.1 Principal Notational Conventions

This section explains notational conventions that are used throughout this dissertation.

The *positive* and *negative parts* of a real number $x$ are denoted by $x^+$ and $x^-$, respectively, that is, $x^+ := \max\{0, x\}, x^- := -\min\{0, x\}$. The *integer part* of $x$ is denoted by $\lfloor x \rfloor$ and $\{x\}$ denotes the *fractional part*, i.e. $\{x\} := x - \lfloor x \rfloor$.

For a random variable (r.v.) $X$, the *distribution function* (d.f.) and *tail function* (t.f.) are denoted by

$$F_X(x) := \mathbb{P}(X \leqslant x) \quad \text{and} \quad \overline{F}_X(x) := \mathbb{P}(X > x).$$

$F^{*n}$ is the *n-fold convolution of a d.f.* $F$, meaning that, if $X_1, \ldots, X_n$ are independent copies of a r.v. $X$, then $F_X^{*n}(x) := \mathbb{P}(X_1 + \ldots + X_n \leqslant x)$. Accordingly, $\overline{F_X^{*n}}(x) := \mathbb{P}(X_1 + \ldots + X_n > x)$. Similarly, $F_X \otimes F_Y$ denotes the *product convolution of a d.f.s* $F_X$ and $F_Y$, meaning that, if $X$ and $Y$ are independent, then $F_X \otimes F_Y(x) := \mathbb{P}(XY \leqslant x)$ and, accordingly, $\overline{F_X \otimes F_Y}(x) := \mathbb{P}(XY > x)$.

For a d.f. $F : \mathbb{R} \to [0, 1]$, the *support of* $F$ is defined as

$$\operatorname{supp} F := \{x : F(x + \delta) - F(x - \delta) > 0 \quad \text{for all } \delta > 0\}$$

$$= \{x : \overline{F}(x - \delta) - \overline{F}(x + \delta) > 0 \quad \text{for all } \delta > 0\}.$$

If $\operatorname{supp} F \subset [0, \infty)$, then it is said that the distribution (or d.f.) $F$ is *supported on* $\mathbb{R}^+$. If $\operatorname{supp} F \not\subset [0, \infty)$, then it is said that the distribution (or d.f.) $F$ is *supported on* $\mathbb{R}$. In addition, it is said that a r.v. $X$ has an *unbounded right tail* if $\overline{F}_X(x) > 0$ for all $x \in \mathbb{R}$. The value $x \in [-\infty, \infty)$ is called the *greatest lower bound* of the r.v. $X$ if $\mathbb{P}(X \geqslant x) = 1$ and $\mathbb{P}(X \geqslant x + \delta) < 1$ for all $\delta > 0$.

The standard notation for *indicator* is used:

$$\mathbb{1}_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

In addition, for two real functions $f(x)$ and $g(x)$ which are positive for sufficiently large $x$, the following notations are used:

$$f(x) \underset{x\to\infty}{\sim} g(x) \qquad \text{if} \qquad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1;$$

$$f(x) \underset{x\to\infty}{\lesssim} g(x) \qquad \text{if} \qquad \limsup_{x\to\infty} \frac{f(x)}{g(x)} \leqslant 1;$$

$$f(x) = o(g(x)) \qquad \text{if} \qquad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0;$$

$$f(x) = O(g(x)) \qquad \text{if} \qquad \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty;$$

$$f(x) \asymp g(x) \qquad \text{if} \qquad f(x) = O(g(x)) \quad \text{and} \quad g(x) = O(f(x)).$$

Note that all limiting relationships and asymptotic estimates, unless stated otherwise, are understood as $x$ tends to infinity, therefore, for example, $f(x) \sim g(x)$ is the same as $f(x) \underset{x\to\infty}{\sim} g(x)$.

To enhance readability, a summary of the notational conventions introduced in this section is also included at the beginning of the dissertation.

## 1.2 Heavy-Tailed Distributions

In probability theory, heavy-tailed distributions play a crucial role when analysing random variables for which extremely large outcomes occur with non-negligible probability. In contrast to light-tailed distributions, whose tails decay at least exponentially and thus possess finite exponential moments, heavy-tailed distributions exhibit a slower rate of decay, leading to the divergence of all exponential moments. In this dissertation, the following definition of heavy-tailed distributions is adopted.

**DEFINITION 1.1.** *A distribution function $F$ supported on $\mathbb{R}$ is said to be heavy-tailed, denoted by $F \in \mathscr{H}$, if for all $h > 0$*

$$\int_{-\infty}^{\infty} e^{hx} \mathrm{d}F(x) = \infty.$$

*Otherwise, $F$ is said to be light-tailed and is denoted by $F \in \mathscr{H}^{c}$.*

Moreover, due to the alternative expectation formula (see, e.g., [74, 95]), the following holds

$$\int_{[0,\infty)} \mathrm{e}^{\delta x} \mathrm{d}F(x) = \overline{F}(-0) + \delta \int_{0}^{\infty} \mathrm{e}^{\delta x} \overline{F}(x)\, \mathrm{d}x$$

implying that

$$F \in \mathscr{H} \Rightarrow \ \overline{F}(x) > 0,\, x \in \mathbb{R};$$

$$F \in \mathscr{H} \Leftrightarrow \limsup_{x \to \infty} \mathrm{e}^{\delta x} \overline{F}(x) = \infty \text{ for each } \delta > 0.$$

As already mentioned in the introduction, although this dissertation is primarily theoretical in nature, it is important to emphasize that the relevance of heavy-tailed distributions extends well beyond purely theoretical considerations. It is often encountered in practical applications, particularly in financial and actuarial risk management, where extreme events — such as catastrophic losses or exceptionally large claims — cannot be regarded as negligible fluctuations. Instead, these events exert a significant influence on aggregate outcomes and the overall assessment of risk. Below an example of such distribution is provided.

**EXAMPLE 1.1.** *It is said that r.v. $X$ is distributed according to the Weibull distribution with some scale $\kappa > 0$ and shape $\alpha > 0$ parameters if its tail function has the following form*

$$\overline{F}(x) = e^{-(\frac{x}{\kappa})^{\alpha}}.$$

*$F \in \mathscr{H}$ if and only if $\alpha < 1$.*

More examples of heavy-tailed distributions can be found in, e.g., [41].

In the literature, several subclasses of heavy-tailed distributions are often distinguished, each capturing a different degree of tail heaviness and structural regularity. Their relationships form a well-known hierarchical structure (see, e.g., [18]), summarized in the figure below.

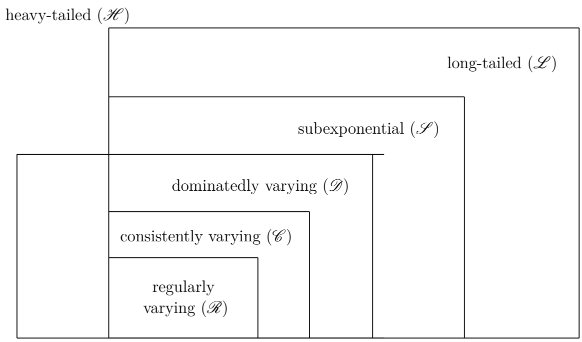


**FIGURE 1.1.** $\mathscr{R} \subset \mathscr{C} \subset \mathscr{D} \cap \mathscr{L} \subset \mathscr{S} \subset \mathscr{L} \subset \mathscr{H}, \mathscr{D} \subset \mathscr{H}$.

The following subsections examine each of these subclasses in greater detail.

### 1.2.1 Long-Tailed Distributions

The class of long-tailed distributions traces its origins to Chistyakov's 1964 paper (see [25]), where the defining tail property was first introduced. The defining feature of these distributions is insensitivity of the tail to bounded perturbations: shifting the argument by any fixed constant leaves the tail essentially unchanged in an asymptotic sense. It captures an idea that extremely large values remain likely even after substantial shifts. In this dissertation, the following definition of long-tailed distributions is adopted.

**DEFINITION 1.2.** *A distribution function $F$ supported on $\mathbb{R}$ is said to be long-tailed, denoted by $F \in \mathscr{L}$, if for any $y > 0$*

$$\overline{F}(x+y) \sim \overline{F}(x).$$

An example of such distribution is given below.

**EXAMPLE 1.2.** *It is said that r.v. $X$ is distributed according to the lognormal distribution with parameters $\mu$ and $\sigma > 0$ if its tail function* *has the following form*

$$\overline{F}(x) = 1 - \Phi\left(\frac{\log x - \mu}{\sigma}\right),$$

*where* $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{t^2}{2}} \mathrm{d}t$ *is a d.f. of a standard normal distribution.*

*It can be easily verified that for any fixed* $y > 0$*,*

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1,$$

*therefore* $F \in \mathscr{L}$*.*

### 1.2.2 Subexponential Distributions

Alongside the class of long-tailed distributions, Chistyakov also introduced the subexponential class (see [25]). The defining feature of these distributions is that adding two independent copies of r. v. from subexponential class produces a tail that is asymptotically twice as large as the original, indicating that extreme values typically arise from a single exceptionally large observation. This ”one large jump” phenomenon provides a natural framework for analysing rare but impactful events in systems where extreme shocks are the primary drivers of aggregate risk. In this dissertation, the following definition of subexponential distributions is adopted.

**DEFINITION 1.3.** *A distribution function* $F$ *supported on* $\mathbb{R}$ *is said to be subexponential, denoted by* $F \in \mathscr{S}$*, if* $F \in \mathscr{L}$ *and* $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$*.*

An example of a distribution belonging to this class is presented below.

**EXAMPLE 1.3.** *It is said that r.v.* $X$ *is distributed according to the Cauchy distribution with some scale* $\kappa > 0$ *and position* $a \in \mathbb{R}$ *parameters if its density function has the following form*

$$f(x) = \frac{\kappa}{\pi((x-a)^2 + \kappa^2)}.$$

*The tail function of Cauchy distribution has the following form*

$$\overline{F}(x) = \frac{1}{\pi} \arctan\left(\frac{\kappa}{x-a}\right)$$

*and the tail of 2-fold convolution is*

$$\overline{F^{*2}}(x) = \frac{1}{\pi} \arctan\left(\frac{2\kappa}{x-2a}\right).$$

*It can be easily verified that* $\overline{F}(x+y) \sim \overline{F}(x)$ *for any fixed* $y > 0$ *and* $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$*, thus* $F \in \mathscr{S}$*.*

**REMARK 1.1.** *It can also be shown that both examples discussed in the previous subsections (Weibull and lognormal distributions) belong to the subexponential class. However, the constructions of Pitman and of Embrechts and Goldie distributions demonstrate that* $\mathscr{L} \setminus \mathscr{S} \neq \emptyset$*. Further details can be found in [35] and [87].*

### 1.2.3 Dominatedly Varying Distributions

The class of dominatedly varying distributions was described more than half a century ago (see [40]). These distributions are characterized by a controlled form of tail behaviour under scaling. The defining feature of these distributions is that scaling the argument down by any fixed factor does not increase the tail probability by more than a constant factor. In this dissertation, the following definition is adopted.

**DEFINITION 1.4.** *A distribution function* $F$ *supported on* $\mathbb{R}$ *is said to be dominatedly varying, denoted by* $F \in \mathscr{D}$*, if for any (equivalently for some)* $y \in (0,1)$

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

A common example (see, e.g., [50]) of dominatedly varying distribution is provided below.

**EXAMPLE 1.4.** *It is said that r. v.* $\mathcal{PP} = \mathcal{PP}_{\{a,b\}}$ *is distributed according to the generalized Peter and Paul law with parameters* $\{a, b\}$*,* $(b > 1, a \in (0,\infty))$ *if*

$$\mathbb{P}\big(\mathcal{PP}_{\{a,b\}} = b^k\big) = (b^a - 1)b^{-ak},\ k \in \mathbb{N} := \{1, 2, \ldots\}.$$

*One can obtain that the tail function has the following form*

$$\overline{F}_{\mathcal{PP}_{\{a,b\}}}(x) = (b^a - 1) \sum_{k \geqslant 1, b^k > x} b^{-ak} = \mathbb{1}_{\{x \in (-\infty, 1)\}} + \left(b^{-a}\right)^{\lfloor \log_b x \rfloor} \mathbb{1}_{\{x \in [1, \infty)\}}$$

*which implies that* $F_{\mathcal{PP}} \in \mathscr{D}$.

### 1.2.4 Consistently Varying Distributions

The class of consistently varying distributions represents a refinement of dominated variation and imposes a stronger form of tail regularity. It was introduced by Cline (see [27]) and was named there as "intermediate regular variation". A distribution belongs to this class if its tail remains stable under scaling. To be more precise, the defining feature of this class is that small multiplicative changes in the argument have an asymptotically negligible effect on the tail. In this dissertation, the following definition is adopted.

**DEFINITION 1.5.** *A distribution function* $F$ *supported on* $\mathbb{R}$ *is said to be consistently varying, denoted by* $F \in \mathscr{C}$, *if*

$$\lim_{y \uparrow 1} \limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

Further, an example satisfying this definition is given below.

**EXAMPLE 1.5.** *It is said that r. v.* $\mathcal{CT} = \mathcal{CT}_r$ *is distributed according to the Cai-Tang law with parameter* $r \in (0,1)$ *if* $\mathcal{CT} = (1 + \mathcal{U})\, 2^{\mathcal{G}}$, *where* $\mathcal{U}$ *and* $\mathcal{G}$ *are independent,* $\mathcal{U}$ *denotes uniform distribution on interval* $[0,1]$ *and* $\mathcal{G}$ *denotes the geometric distribution with parameter* $r \in (0,1)$, *that is,* $\mathbb{P}(\mathcal{G} = k) = (1-r)r^k, k \in \mathbb{N}_0 := \{0, 1, \ldots\}$.

*For* $x \geqslant 1$ *the distribution function has the following form*

$$F_{\mathcal{CT}_r}(x) = (1-r) \left( \frac{1 - r^{\lfloor \log_2 x \rfloor}}{1-r} + \left( \frac{x}{2^{\lfloor \log_2 x \rfloor}} - 1 \right) r^{\lfloor \log_2 x \rfloor} \right).$$

*The above expression implies that* $F_{\mathcal{CT}_r} \in \mathscr{C}$ *for all* $r \in (0,1)$.

**REMARK 1.2.** *For the Peter and Paul distribution (see Example 1.4),* *it can be derived that*

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}_{\mathcal{PP}_{\{a,b\}}}(xy)}{\overline{F}_{\mathcal{PP}_{\{a,b\}}}(x)} = b^a \neq 1,$$

*which is a common example of a distribution belonging to the class $\mathscr{D}$ but not to the class $\mathscr{C}$, proving that $\mathscr{D}\setminus\mathscr{C}\neq\emptyset$.*

### 1.2.5 Regularly Varying Distributions

The class of regularly varying distributions was introduced by Karamata in the early 1930s and is characterized by tails that behave like a power function at infinity. The defining feature of this class is that scaling the argument by a constant factor changes the tail probability only by a predictable power of that factor. In this dissertation, the following definition of regularly varying class is adopted.

**DEFINITION 1.6.** *A distribution function $F$ supported on $\mathbb{R}$ is said to be regularly varying with index $\alpha\geqslant 0$, denoted by $F\in\mathscr{R}_\alpha$, if for any $y>0$*

$$\lim_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}.$$

*The class of all regularly varying distributions is $\mathscr{R}=\bigcup\limits_{\alpha\geqslant 0}\mathscr{R}_\alpha$.*

Below an example of such distribution is provided.

**EXAMPLE 1.6.** *It is said that r. v. $\mathcal{P}a=\mathcal{P}a_{\{a,b\}}$ is distributed according to the Pareto distribution with parameters $a,b>0$ if its d. f. has the form*

$$F_{\mathcal{P}a_{\{a,b\}}}(x) = \begin{cases} 1-(1+\frac{x}{b})^{-a} & , x\geqslant 0,\\ 0 & , x<0.\end{cases}$$

*It is obvious that $\overline{F}(x)=1-F(x)=(1+\frac{x}{b})^{-a}\sim(\frac{x}{b})^{-a}$.*

*From the last expression it is easy to observe that, for all $b>0$, $F\in\mathscr{R}_\alpha$ where $\alpha=a$ (the first parameter of the Pareto distribution).*

**REMARK 1.3.** *It can be shown that Cai-Tang distribution (see Example 1.5) does not belong to regularly varying class, proving that $\mathscr{C}\setminus\mathscr{R}\neq\emptyset$.*

*For more details see [18].*

In the literature, a direct generalization of the class $\mathscr{R}$ is often considered and is further explored in this dissertation. The class of extended regularly varying distributions was introduced by Matuszewska and Orlicz (see [79]) and allows the tail exponent to vary slowly with the argument while maintaining controlled scaling behaviour. In this dissertation, the following definition is adopted.

**DEFINITION 1.7.** *A distribution function $F$ supported on $\mathbb{R}$ is said to be extended regularly varying with indices $0 \leqslant \alpha \leqslant \beta < \infty$, denoted by $F \in \mathscr{ERV}_{\{\alpha,\beta\}}$, if for any $y > 1$*

$$y^{-\beta} \leqslant \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant y^{-\alpha}.$$

*The class of all extended regularly varying distributions is*

$$\mathscr{ERV} = \bigcup_{0\leqslant\alpha\leqslant\beta<\infty} \mathscr{ERV}_{\{\alpha,\beta\}}.$$

**REMARK 1.4.** *It holds that $\mathscr{R} \subset \mathscr{ERV} \subset \mathscr{C}$, notice that both inclusions in this relation are proper. Due to results of [28], the distribution with t. f.*

$$\overline{F}(x) = e^{-\lfloor \log(1+x)\rfloor + \left(\log(1+x) - \lfloor \log(1+x)\rfloor\right)^{\frac{1}{2}}}, \quad x \geqslant 0,$$

*belongs to $\mathscr{C} \setminus \mathscr{ERV}$, and the distribution with t. f.*

$$\overline{F}(x) = e^{-\lfloor \log(1+x)\rfloor + \left(\log(1+x) - \lfloor \log(1+x)\rfloor\right)^{2}}, \quad x \geqslant 0,$$

*belongs to $\mathscr{ERV} \setminus \mathscr{R}$.*

### 1.2.6 Auxiliary Indices

This subsection recalls two indices used to assess whether a distribution $F$ belongs to the previously defined classes.

The first index is the upper Matuszewska index, introduced in [78]. In this dissertation, a slightly different but equivalent formulation is adopted, following [12], which has been widely used in subsequent literature (e.g. [22, 58, 102, 111, 135, 137]).

**DEFINITION 1.8.** *For a distribution function $F$ the upper Matuszewska index $J_F^+$ is defined by equality*

$$J_F^+ := \inf_{y>1} \left\{ -\frac{1}{\log y} \log \left\{ \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right\} \right\}.$$

The second index is the so-called L-index, which appears in a number of studies (e.g. [33, 58, 72, 118, 135]).

**DEFINITION 1.9.** *The L-index for a distribution function $F$ is defined by equality*

$$L_F := \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

The aforementioned indices give an important characterization for heavy-tailed distribution functions (see, e.g., [118]):

$$F \in \mathscr{D} \quad \Leftrightarrow \quad L_F > 0 \quad \Leftrightarrow \quad J_F^+ < \infty;$$

$$F \in \mathscr{C} \quad \Leftrightarrow \quad L_F = 1 \quad \Rightarrow \quad J_F^+ < \infty;$$

$$F \in \mathscr{R}_\alpha \quad \Rightarrow \quad L_F = 1, \quad J_F^+ = \alpha.$$

## 1.3 QAI Dependence Structure

The assumption of independence between random variables is often too restrictive in both theoretical research and practical applications. Consequently, many authors have considered dependence presumption of some sort while investigating the asymptotic behaviour of random sums with heavy-tailed summands. The main dependence assumption adopted in this dissertation is the notion of pairwise quasi-asymptotically independence, introduced by Chen and Yuen in [22] and subsequently considered in [48, 73, 75, 109, 110, 123], among others. This concept provides a flexible framework for analysing sums of heavy-tailed r.v.s that are not required to be independent but whose dependence becomes negligible in the tail. In this dissertation, the following definition is adopted.

**DEFINITION 1.10.** *Real-valued random variables $\{\xi_1, \ldots, \xi_n\}$ with distribution functions supported on $\mathbb{R}$ are called pairwise quasi- asymptotically independent (pQAI) if for any pair with indices $k, l \in \{1, \ldots, n\}$, $k \neq l$,*

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(\xi_k^+ > x, \xi_l^+ > x\right)}{\overline{F}_{\xi_k^+}(x) + \overline{F}_{\xi_l^+}(x)} = \lim_{x\to\infty} \frac{\mathbb{P}\left(\xi_k^+ > x, \xi_l^- > x\right)}{\overline{F}_{\xi_k^+}(x) + \overline{F}_{\xi_l^+}(x)} = 0.$$

One of the ways to construct collections of pQAI r.v.s is by choosing a specific copula, which, by Sklar's theorem (see [81, 93]), then allows to obtain a multivariate distribution satisfying the above definition. Below the definition of a two-dimensional copula is recalled.

**DEFINITION 1.11.** *A two-dimensional copula is a function $C : [0,1]^2 \to [0,1]$ which satisfies the following properties:*

1. $C(x,0) = C(0,x) = 0,$ *for all* $x \in [0,1];$
2. $C(x,1) = C(1,x) = x,$ *for all* $x \in [0,1];$
3. *for all* $0 \leq u_1 \leq u_2 \leq 1$ *and* $0 \leq v_1 \leq v_2 \leq 1,$

$$C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0.$$

Known examples of copulas allowing to construct pQAI r.v.s include Farley-Gumbel-Morgenstein (FGM) and Ali-Michail-Haq (AMH), see more detailed examples below.

**EXAMPLE 1.7.** *Let $\xi_1, \xi_2$ be r.v.s supported on $\mathbb{R}$ and with corresponding marginal d.f.s $F_1, F_2$. Consider the two-dimensional FGM copula*

$$C_\vartheta(u_1, u_2) = u_1 u_2 + \vartheta u_1 u_2 (1 - u_1)(1 - u_2), \quad u_1, u_2 \in [0,1], \quad \vartheta \in [-1,1].$$

*Let r.v.s $\xi_1, \xi_2$ have a joint d.f. $C_\vartheta(F_1(x_1), F_2(x_2))$ with some $\vartheta \in [-1,1]$. It follows from Sklar's theorem (see, for instance, Theorem 2.3.3 in [81]) that for any given marginal d.f.s $F_1, F_2$ and an arbitrary copula $C(u_1, u_2)$, the function $F(x_1, x_2) := C(F_1(x_1), F_2(x_2))$ is a bivariate d.f. with marginal d.f.s $F_1, F_2$. Then*

$$\frac{\mathbb{P}(\xi_1 > x, \xi_2 > x)}{\overline{F}_1(x) + \overline{F}_2(x)} = \frac{1 - F_1(x) - F_2(x) + C_\vartheta(F_1(x), F_2(x))}{\overline{F}_1(x) + \overline{F}_2(x)}$$

$$= \frac{\overline{F}_1(x)\overline{F}_2(x)(1+\vartheta F_1(x)F_2(x))}{\overline{F}_1(x)+\overline{F}_2(x)} \leqslant 2\overline{F}_1(x).$$

*Similarly, by observing that*

$$\mathbb{P}(\xi_1 > x, \xi_2^- > x) \leqslant \mathbb{P}(\xi_1 > x, \xi_2^- \geqslant x)$$

$$= \mathbb{P}(\xi_2 \leqslant -x) - \mathbb{P}(\xi_1 \leqslant x, \xi_2 \leqslant -x)$$

*for positive $x$, it can be derived that*

$$\frac{\mathbb{P}(\xi_1 > x, \xi_2^- > x)}{\overline{F}_1(x)+\overline{F}_2(x)} \leqslant \frac{F_2(-x) - C_\vartheta(F_1(x), F_2(-x))}{\overline{F}_1(x)+\overline{F}_2(x)}$$

$$= \frac{\overline{F}_1(x)F_2(-x)(1-\vartheta F_1(x)F_2(-x))}{\overline{F}_1(x)+\overline{F}_2(x)} \leqslant 2F_2(-x).$$

*Hence $\xi_1, \xi_2$ are QAI.*

**REMARK 1.5.** *In a more general setting, it could be proven that $n$-dimensional ($n \geqslant 2$) FGM distributions (see, e.g., [66]) are also pQAI. For more details, see [33].*

**EXAMPLE 1.8.** *Let $\xi_1, \xi_2$ be r.v.s with corresponding d.f.s $F_1, F_2$. Consider random vector $(\xi_1, \xi_2)$ with a bivariate d.f.*

$$F(x_1, x_2) := C_\vartheta(F_1(x_1), F_2(x_2)),$$

*where $C_\vartheta$ is the AMH copula [4]:*

$$C_\vartheta(u_1, u_2) = \frac{u_1 u_2}{1 - \vartheta(1-u_1)(1-u_2)}, \quad u_1, u_2 \in [0,1], \quad \vartheta \in (-1, 1).$$

*Following the same idea as in the previous example, for positive $x$, it can be shown that*

$$\frac{\mathbb{P}(\xi_1 > x, \xi_2 > x)}{\overline{F}_1(x)+\overline{F}_2(x)} = \frac{1 - F_1(x) - F_2(x) + C_\vartheta(F_1(x), F_2(x))}{\overline{F}_1(x)+\overline{F}_2(x)}$$

$$= \frac{\overline{F}_1(x)\overline{F}_2(x)(1+\vartheta(F_2(x)-\overline{F}_1(x)))}{(\overline{F}_1(x)+\overline{F}_2(x))(1-\vartheta\overline{F}_1(x)\overline{F}_2(x))} \leqslant \frac{2\overline{F}_2(x)}{1-\vartheta\overline{F}_1(x)\overline{F}_2(x)}$$

*and*

$$\frac{\mathbb{P}(\xi_1 > x, \xi_2^- > x)}{\overline{F}_1(x)+\overline{F}_2(x)} \leqslant \frac{F_2(-x) - C_\vartheta(F_1(x), F_2(-x))}{\overline{F}_1(x)+\overline{F}_2(x)}$$

$$= \frac{F_2(-x)\overline{F}_1(x)(1-\vartheta\overline{F}_2(-x))}{(\overline{F}_1(x)+\overline{F}_2(x))(1-\vartheta\overline{F}_1(x)\overline{F}_2(-x))} \leqslant \frac{2F_2(-x)}{1-\vartheta\overline{F}_1(x)\overline{F}_2(-x)}.$$

*Hence $\xi_1, \xi_2$ are QAI.*

For further discussion on the application of copulas in modelling dependence for heavy-tailed distributions, see [2, 7, 33, 38, 81, 114, 130] and the references therein.

In the literature, similar dependence structures have also been considered. For instance, Geluk and Tang in [48] assumed that r.v.s $\xi_1, \ldots, \xi_n$ satisfy the so-called Assumption A, which was later referred to as strong quasi-asymptotic independence (see, e.g., [58, 72, 73, 110]). The definition is recalled below.

**DEFINITION 1.12.** *Random variables $\{\xi_1, , \ldots, \xi_n\}$ with unbounded right tails are called pairwise strongly quasi-asymptotically independent (pSQAI) if for any $k, l \in \{1, \ldots, n\}$, $k \neq l$,*

$$\lim_{\min\{x_k, x_l\}\to\infty} \mathbb{P}(|\xi_k| > x_k | \xi_l > x_l) = 0.$$

Moreover, Cheng in [23] considered two other dependence structures, namely pairwise tail quasi-asymptotic independence and pairwise asymptotic independence. The corresponding definitions are given below.

**DEFINITION 1.13.** *Random variables $\{\xi_1, \ldots, \xi_n\}$ with unbounded right tails are called pairwise tail quasi-asymptotically independent (pTQAI) if for any $k, l \in \{1, \ldots, n\}$, $k \neq l$,*

$$\lim_{\min\{x_k, x_l\}\to\infty} \frac{\mathbb{P}(\xi_k^+ > x_k, \xi_l^+ > x_l)}{\overline{F}_{\xi_k^+}(x_k) + \overline{F}_{\xi_l^+}(x_l)}$$

$$= \lim_{\min\{x_k, x_l\}\to\infty} \frac{\mathbb{P}(\xi_k^- > x_k, \xi_l^+ > x_l)}{\overline{F}_{\xi_k^+}(x_k) + \overline{F}_{\xi_l^+}(x_l)} = 0.$$

**DEFINITION 1.14.** *Random variables $\{\xi_1, \ldots, \xi_n\}$ with unbounded right tails are called pairwise asymptotically independent (pAI) if for any $k, l \in \{1, \ldots, n\}$, $k \neq l$,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\overline{F}_{\xi_k}(x)} = \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\overline{F}_{\xi_l}(x)} = 0.$$

It is worth noting that all of the dependence structures described above imply pQAI. For further details, see, e.g., [33].

## 1.4 Risk Measures

Let a real-valued r.v. $X$ with d.f. $F_X$ represent a random risk associated with an uncertain financial position, insurance loss, or other random outcome. A natural objective is then to quantify the magnitude of this risk. Risk measures provide a systematic framework for assessing and comparing such risks. The study of risk measures is a relatively popular topic in probability theory, e.g., classical mathematical problems related to risk measures are discussed in [90], distortion risk measures are considered in [5, 85], the duality of risk measures is considered in [67], risk measures with special properties are analysed in [59, 113, 127], and risk measures constructed using higher-order moments are analysed in [19, 53, 77, 86].

In probabilistic terms, a risk measure assigns a real number to a random variable representing future losses, thereby reducing a potentially complex distributional object to a single interpretable quantity. In this dissertation, the following definition of a risk measure is adopted.

**DEFINITION 1.15.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let set $L_0 = \{X \mid X : \Omega \to \mathbb{R}\}$ be a class of real-valued random variables on this space. Then function $\rho : L_0 \to \mathbb{R}$ is called a risk measure.*

Note that throughout this dissertation, unless stated otherwise, positive value of $X$ corresponds to loss, while negative value represents profit. Further, several classical examples of risk measures are provided.

**EXAMPLE 1.9.** *The simplest example of a risk measure is the expectation $\mathbb{E}X$. However, it often fails to adequately capture tail risk and may be infinite for heavy-tailed distributions.*

*Another commonly used risk measure is Value-at-Risk (VaR), defined as*

$$\mathrm{VaR}_q(X) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(X > x) \leqslant 1 - q\}.$$

*While VaR specifies a quantile of the loss distribution, it does not take into account the magnitude of losses beyond this threshold and fails to* *adequately reflect diversification effects. Therefore, it may underestimate risk in market stress situations under extreme asset price fluctuations (see [126]).*

*A further widely used risk measure is the Expected Shortfall (ES), defined as*

$$\mathrm{ES}_q(X) = \frac{1}{1-q} \int_q^1 \mathrm{VaR}_u(X)\mathrm{d}u.$$

*ES addresses some of the limitations of VaR by incorporating information about the severity of losses in the tail.*

As noted in the example above, VaR may fail to adequately reflect diversification effects. This limitation motivated the introduction of coherent risk measures by Artzner et al. (see [6]), which ensures a meaningful economic interpretation. The definition of such risk measure is recalled below.

**DEFINITION 1.16.** *Risk measure $\rho : L_0 \to \mathbb{R}$ is called coherent if it satisfies the following axioms:*

(i) *Monotonicity:*

$$\rho(X) \leqslant \rho(Y), \quad \textit{if} \quad X \leqslant Y \ \textit{almost surely}, \qquad X, Y \in L_0;$$

(ii) *Translation Invariance:*

$$\rho(X + a) = \rho(X) + a, \quad \textit{for all} \quad X \in L_0, \quad a \in \mathbb{R};$$

(iii) *Positive Homogeneity:*

$$\rho(\lambda X) = \lambda \rho(X), \quad \textit{for all} \quad X \in L_0, \quad \lambda \geq 0;$$

(iv) *Subadditivity:*

$$\rho(X + Y) \leqslant \rho(X) + \rho(Y), \quad \textit{for all} \quad X, Y \in L_0.$$

Note that ES is a coherent risk measure (see, e.g., [1]). Another example is the Haezendonck-Goovaerts (HG) risk measure which is a direct generalization of the ES and was originally introduced in [54]. According to the results of [10, 51], this measure has many properties necessary for risk measurements. As already mentioned, it is coherent and, as a result, is a monetary and convex risk measure as in [43, 45]. HG risk measure has attracted much attention in recent years (see, e.g., [10, 105, 106, 115]) and constitutes one of the main focuses of this dissertation. Since the HG risk measure is defined on the Orlicz heart, the notions of a normalized Young function, an Orlicz space, and the Orlicz heart are recalled prior to introducing the definition of the HG risk measure.

**DEFINITION 1.17.** *A function $\varphi$ defined on $\mathbb{R}$ is said to be normalized Young function if $\varphi$ is non-negative, convex on interval $[0, \infty)$ such that $\varphi(0) = 0, \varphi(1) = 1$, $\varphi(\infty) = \infty$.*

**DEFINITION 1.18.** *Orlicz space $\mathbb{L}^\varphi$ and Orlicz heart $\mathbb{L}_0^\varphi$ of real-valued r.v.s $X$ associated with function $\varphi$ are defined by the following equalities:*

$$\mathbb{L}^\varphi = \Big\{X : \mathbb{E}\big(\varphi(cX)\big) < \infty \text{ for some } c > 0\Big\},$$

$$\mathbb{L}_0^\varphi = \Big\{X : \mathbb{E}\big(\varphi(cX)\big) < \infty \text{ for all } c > 0\Big\}.$$

Note that $\mathbb{L}^\varphi = \mathbb{L}_0^\varphi$ if

$$\limsup_{x\to\infty} \frac{\varphi(2x)}{\varphi(x)} < \infty.$$

**DEFINITION 1.19.** *Let $\varphi$ be a Young function, $q \in (0, 1)$, and $X \in \mathbb{L}_0^\varphi$. The Haezendonck-Goovaerts (HG) risk measure for variable $X$ is defined by equality*

$$HG_q(X) = \inf_{x\in\mathbb{R}} \big(x + h(x, q)\big), \tag{1.1}$$

*where $h = h(x, q)$ is a solution of the equation*

$$\mathbb{E}\varphi\left(\frac{(X - x)^+}{h}\right) = 1 - q$$

*if $\overline{F}_X(x) > 0$, and $h(x, q) = 0$ if $\overline{F}_X(x) = 0$.*

According to Proposition 3(b,d) in [10], the minimizer $(x_*, h_*)$ to equation (1.1) exists for all $q \in (0, 1)$, and it is unique if the function $\varphi$ is strictly convex.

As already mentioned, the HG risk measure is a direct generalisation of the ES risk measure obtained by taking the following Young function $\varphi(t) = t$. Other generalisations of the ES measure are obtained by replacing quantiles with generalised quantiles or expectiles. Such risk measures are described and studied in [9, 11, 138].

# 2 GENERALIZED MOMENTS OF SUMS WITH HEAVY-TAILED RANDOM SUMMANDS

For a fixed $n \in \mathbb{N}$, consider two independent collections:

(i) $\{\xi_1, \dots, \xi_n\}$ of real-valued, possibly dependent, heavy-tailed r.v.s, called primary r.v.s;
(ii) $\{\theta_1, \dots, \theta_n\}$ of nonnegative, nondegenerate at zero r.v.s, called random weights.

This chapter is devoted to the study of the asymptotic behaviour of sums of primary r.v.s

$$S_n^\xi := \sum_{k=1}^{n} \xi_k = \xi_1 + \ldots + \xi_n,$$

and randomly weighted sums

$$S_n^{\theta\xi} := \sum_{k=1}^{n} \theta_k \xi_k = \theta_1 \xi_1 + \ldots + \theta_n \xi_n.$$

The asymptotic behaviour of both types of sums has been extensively studied in recent years within the applied probability literature, under various assumptions on the collections $\{\xi_1, \dots, \xi_n\}$, $\{\theta_1, \dots, \theta_n\}$, and their dependence structures. Accordingly, this chapter begins with a review of known results on the asymptotic behaviour of tail probabilities $\mathbb{P}(S_n^\xi > x)$ and $\mathbb{P}(S_n^{\theta\xi} > x)$, as well as tail expectations $\mathbb{E}(S_n^\xi \mathbb{1}_{\{S_n^\xi > x\}})$ and $\mathbb{E}(S_n^{\theta\xi} \mathbb{1}_{\{S_n^{\theta\xi} > x\}})$ (Section 2.1). Subsequently, the new results on the asymptotic bounds for the tail expectations $\mathbb{E}(\varphi(S_n^\xi) \mathbb{1}_{\{S_n^\xi > x\}})$ and $\mathbb{E}(\varphi(S_n^{\theta\xi}) \mathbb{1}_{\{S_n^{\theta\xi} > x\}})$ are presented (Section 2.2), followed by proofs of the obtained results (Section 2.3) and several illustrative examples (Section 2.4).

## 2.1 Background

This section begins with a brief review of results from the literature on the asymptotic behaviour of tail probabilities of random and randomly weighted sums.

Although the literature often assumes that the primary r.v.s $\xi_1, \dots, \xi_n$

are independent or identically distributed (see, e.g., [52, 102, 116, 117]), alternative results under less restrictive conditions have also been established. For example, Geluk and Tang (see Theorem 3.1 in [48]) achieved the following:

**THEOREM 2.1.** *Let $\xi_1, \dots, \xi_n$ be real-valued r.v.s with d.f.s $F_{\xi_1}, \dots, F_{\xi_n}$, respectively. If $F_{\xi_k} \in \mathscr{D} \cap \mathscr{L}$ for all $1 \leqslant k \leqslant n$ and $\xi_1, \dots, \xi_n$ are pSQAI, then*

$$\mathbb{P}(S_n^{\xi} > x) \sim \sum_{k=1}^{n} \overline{F}_{\xi_k}(x). \tag{2.1}$$

Concurrently, Chen and Yuen [22] demonstrated that asymptotic relation (2.1) remains valid in the smaller class $\mathscr{C}$ but replacing pSQAI assumption with the weaker pQAI condition (see Theorem 3.1 in [22]). Moreover, in the same article, these results were extended to tail probabilities of randomly weighted sums (see Theorem 3.2 in [22]):

**THEOREM 2.2.** *Let $\xi_1, \dots, \xi_n$ be pQAI real-valued r.v.s such that $F_{\xi_1} \in \mathscr{C}, \dots, F_{\xi_n} \in \mathscr{C}$, and let $\theta_1, \dots, \theta_n$ be nonnegative r.v.s independent of $\xi_1, \dots, \xi_n$ such that*

$$\max\{\mathbb{E}\theta_1^p, \dots, \mathbb{E}\theta_n^p\} < \infty \quad \text{for some} \quad p > \max\{J^+_{F_{\xi_1}}, \dots, J^+_{F_{\xi_n}}\}. \tag{2.2}$$

*Then*

$$\mathbb{P}(S_n^{\theta\xi} > x) \sim \sum_{k=1}^{n} \overline{F}_{\theta_k\xi_k}(x). \tag{2.3}$$

Subsequently, Yi et al. [137] extended the result given in Theorem 2.2 by deriving asymptotic bounds for the case where r.v.s $\xi_1, \dots, \xi_n$ are pQAI and belong to the class $\mathscr{D}$ with an additional tail condition (see Theorem 1 in [137]). These results were later reconsidered and proved by Jaunė et al. [58], who removed that additional tail assumption (see Theorem 1 in [58]):

**THEOREM 2.3.** *Let $\xi_1, \dots, \xi_n$ be pQAI real-valued r.v.s such that $F_{\xi_1} \in \mathscr{D}, \dots, F_{\xi_n} \in \mathscr{D}$, and let $\theta_1, \dots, \theta_n$ be arbitrarily dependent, nonnegative, nondegenerate at zero r.v.s independent of $\xi_1, \dots, \xi_n$ and satisfying* (2.2)*. Then*

$$L_n^{\xi} \sum_{k=1}^{n} \overline{F}_{\theta_k\xi_k}(x) \lesssim \mathbb{P}(S_n^{\theta\xi} > x) \lesssim \frac{1}{L_n^{\xi}} \sum_{k=1}^{n} \overline{F}_{\theta_k\xi_k}(x), \tag{2.4}$$

*where* $L_n^\xi = \min\{L_{F_{\xi_1}}, \dots, L_{F_{\xi_n}}\}$ *and* $L_{F_{\xi_k}} = \lim_{y\downarrow 1}\liminf_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}$ *is the L-index for a d.f.* $F_{\xi_k}$, $k \in \{1,\dots,n\}$.

Moreover, Cheng [23] tightened the bounds in (2.4) by moving the L-indices inside sums and weakened condition (2.2) (see Theorems 1.1 and 1.2 in [23]):

**THEOREM 2.4.** *Let* $\xi_1,\dots,\xi_n$ *be pTQAI real-valued r.v.s such that* $F_{\xi_1} \in \mathscr{D}, \dots, F_{\xi_n} \in \mathscr{D}$, *and let* $\theta_1,\dots,\theta_n$ *be nonnegative r.v.s independent of* $\xi_1,\dots,\xi_n$ *such that*

$$\lim_{x\to\infty}\frac{\overline{F}_{\theta_k}(ux)}{\overline{F}_{\theta_k\xi_k}(x)} = 0 \quad \text{for any } u > 0 \text{ and for all } k \in \{1,\dots,n\}.$$

*Then*

$$\sum_{k=1}^n L_{\xi_k}\overline{F}_{\theta_k\xi_k}(x) \lesssim \mathbb{P}(S_n^{\theta\xi} > x) \lesssim \sum_{k=1}^n \frac{1}{L_{\xi_k}}\overline{F}_{\theta_k\xi_k}(x). \tag{2.5}$$

**REMARK 2.1.** *The same bounds* (2.5) *hold when the pTQAI condition for r.v.s* $\xi_1,\dots,\xi_n$ *is replaced by the pAI condition with additional requirement that*

$$\lim_{x\to\infty}\frac{\overline{F}_{\xi_k^-}(x)}{\overline{F}_{\xi_k}(x)} = 0 \quad \text{for all} \quad k \in \{1,\dots,n\}.$$

Finally, Leipus et al. [68] and Dirma et al. [33] obtained results similar to those in Theorem 2.4, but under a less restrictive pQAI dependence assumption. An exception is that in [33], unlike in [68], the results were formulated for the weighted sum (see Remark 2 in [33]):

**THEOREM 2.5.** *Let* $\xi_1,\dots,\xi_n$ *be pQAI real-valued r.v.s such that* $F_{\xi_1} \in \mathscr{D}, \dots, F_{\xi_n} \in \mathscr{D}$, *and let* $\theta_1,\dots,\theta_n$ *be arbitrarily dependent, nonnegative, nondegenerate at zero r.v.s independent of* $\xi_1,\dots,\xi_n$ *and satisfying* (2.2). *Then*

$$\sum_{k=1}^n L_{\xi_k}\overline{F}_{\theta_k\xi_k}(x) \lesssim \mathbb{P}(S_n^{\theta\xi} > x) \lesssim \sum_{k=1}^n \frac{1}{L_{\xi_k}}\overline{F}_{\theta_k\xi_k}(x). \tag{2.6}$$

The next part of this section provides a brief review of results on the asymptotic behaviour of tail expectations of random and randomly weighted sums which is the main object of this chapter. As noted earlier, primary r.v.s are often assumed to be independent or identically distributed (see, e.g., [107, 131]). However, the focus below is on results obtained under weaker conditions.

Alongside Theorem 2.3, Jaunė et al. [58] established the following results (see Theorem 2 in [58]):

**THEOREM 2.6.** *Let the conditions in Theorem 2.3 be satisfied.*

(i) *If* $\overline{F}_{\theta_k\xi_k}(x) = O(\overline{F}_{\theta_l\xi_l}(x))$ *for all* $k \in \{1,\ldots,n\}$ *and some* $l \in \{1,\ldots,n\}$ *under the condition* $\mathbb{E}\theta_l|\xi_l| < \infty$, *then*

$$\mathbb{E}(\theta_l\xi_l\mathbb{1}_{\{S_n^{\theta\xi}>x\}}) \lesssim \mathbb{E}(\theta_l\xi_l\mathbb{1}_{\{\theta_l\xi_l>x\}}),$$

*and, consequently,*

$$\mathbb{E}(S_n^{\theta\xi}\mathbb{1}_{\{S_n^{\theta\xi}>x\}}) \lesssim \sum_{k=1}^{n}\mathbb{E}(\theta_k\xi_k\mathbb{1}_{\{\theta_k\xi_k>x\}})$$

*if* $\mathbb{E}\theta_1|\xi_1| < \infty$ *and* $\overline{F}_{\theta_k\xi_k}(x) \asymp \overline{F}_{\theta_1\xi_1}(x)$ *for all* $k \in \{1,\ldots,n\}$.

(ii) *If r.v.s* $\xi_1,\ldots,\xi_n$ *are pSQAI and* $\overline{F}_{\theta_k\xi_k}(x) = O(\overline{F}_{\theta_l\xi_l}(x))$ *for all* $k \in \{1,\ldots,n\}$ *and some* $l \in \{1,\ldots,n\}$ *under the condition* $\mathbb{E}\theta_l|\xi_l| < \infty$, *then*

$$L_{F_{\xi_l}}\mathbb{E}(\theta_l\xi_l\mathbb{1}_{\{\theta_l\xi_l>x\}}) \lesssim \mathbb{E}(\theta_l\xi_l\mathbb{1}_{\{S_n^{\theta\xi}>x\}}),$$

*and, consequently,*

$$L_n^{\xi}\sum_{k=1}^{n}\mathbb{E}(\theta_k\xi_k\mathbb{1}_{\{\theta_k\xi_k>x\}}) \lesssim \mathbb{E}(S_n^{\theta\xi}\mathbb{1}_{\{S_n^{\theta\xi}>x\}})$$

*if* $\mathbb{E}\theta_1|\xi_1| < \infty$ *and* $\overline{F}_{\theta_k\xi_k}(x) \asymp \overline{F}_{\theta_1\xi_1}(x)$ *for all* $k \in \{1,\ldots,n\}$.

Subsequently, Leipus et al. [72] considered a dependence structure more restrictive than pQAI, referred to as Assumption B, and extended the above results to tail moments of random sums under the transformation $\varphi(x) = x^{\alpha}$, $\alpha \in \mathbb{N}$. In addition, a corrective constant was introduced in the upper bound (see Theorem 5 in [72]):

**THEOREM 2.7.** *Let* $\xi_1,\ldots,\xi_n$ *be real-valued r.v.s satisfying Assumption B such that* $F_{\xi_1} \in \mathscr{D}$ *and* $\mathbb{E}|\xi_1|^m < \infty$ *for some* $m \in \mathbb{N}$. *Let* $\theta_1,\ldots,\theta_n$ *be nonnegative, nondegenerate at zero, bounded r.v.s independent of* $\xi_1,\ldots,\xi_n$. *If* $\overline{F}_{\theta_k\xi_k}(x) = O(\overline{F}_{\theta_1\xi_1}(x))$ *and* $\overline{F}_{\theta_k\xi_k^-}(x) = O(\overline{F}_{\theta_1\xi_1}(x))$ *for* *all $k \in \{2, \dots, n\}$, then*

$$\begin{aligned} L_n^{\xi} \sum_{k=1}^{n} \mathbb{E}\left((\theta_k \xi_k)^m \mathbb{1}_{\{\theta_k \xi_k > x\}}\right) &\lesssim \mathbb{E}\left((S_n^{\theta\xi})^m \mathbb{1}_{\{S_n^{\theta\xi} > x\}}\right) \\ &\lesssim \frac{1}{L_n^{\xi}} \sum_{k=1}^{n} \mathbb{E}\left((\theta_k \xi_k)^m \mathbb{1}_{\{\theta_k \xi_k > x\}}\right), \end{aligned} \tag{2.7}$$

*where $L_n^{\xi} = \min\{L_{F_{\xi_1}}, \dots, L_{F_{\xi_n}}\}$.*

*Here Assumption B means that r.v.s $\xi_1, \dots, \xi_n$ have unbounded right tails and satisfy*

$$\lim_{x \to \infty} \sup_{u \geqslant x} \mathbb{P}(\xi_k^+ > x | \xi_l^+ > u) = \lim_{x \to \infty} \sup_{u \geqslant x} \mathbb{P}(\xi_k^- > x | \xi_l^+ > u)$$

$$= \lim_{x \to \infty} \sup_{u \geqslant x} \mathbb{P}(\xi_k^+ > x | \xi_l^- > u) = 0$$

*for all $k, l \in \{1, \dots, n\}$, $k \neq l$.*

Note that similar bounds to (2.7) were obtained for tail moments of unweighted sums (see Theorem 4 in [72]).

Bounds (2.7) were later sharpened for sums of primary r.v.s by Leipus et al. [68], where individual $L$-indices, $L_{F_{\xi_k}}$, $k \in \{1, \dots, n\}$, were put inside the bounding sums. In addition, the range of possible $\alpha$ values was extended to $\mathbb{N}_0$. Finally, Dirma et al. [33] considered a less restrictive pQAI dependence structure and further generalized these results to randomly weighted sums and allowed $\alpha$ to take any nonnegative real value (see Theorem 4 in [33]):

**THEOREM 2.8.** *Let $\xi_1, \dots, \xi_n$ be pQAI real-valued r.v.s such that $F_{\xi_1} \in \mathscr{D}, \dots, F_{\xi_n} \in \mathscr{D}$, and let $\theta_1, \dots, \theta_n$ be arbitrarily dependent, nonnegative, nondegenerate at zero r.v.s satisfying* (2.2). *If collections $\{\xi_1, \dots, \xi_n\}$ and $\{\theta_1, \dots, \theta_n\}$ are independent and $\mathbb{E}(\theta_k |\xi_k|)^{\alpha} < \infty$ for all $k \in \{1, \dots, n\}$ and some $\alpha \in [0, \infty)$, then*

$$\begin{aligned} \sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\left((\theta_k \xi_k)^{\alpha} \mathbb{1}_{\{\theta_k \xi_k > x\}}\right) &\lesssim \mathbb{E}\left((S_n^{\theta\xi})^{\alpha} \mathbb{1}_{\{S_n^{\theta\xi} > x\}}\right) \\ &\lesssim \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\left((\theta_k \xi_k)^{\alpha} \mathbb{1}_{\{\theta_k \xi_k > x\}}\right). \end{aligned} \tag{2.8}$$

Additionally, three remarks were made in [33] which are also provided below.

**REMARK 2.2.** *By narrowing the class $\mathscr{D}$ to the class $\mathscr{C}$ and noticing that $F \in \mathscr{C} \Leftrightarrow L_F = 1$, the exact asymptotic equivalence could be obtained. In particular, if $F_{\xi_k} \in \mathscr{C}$ for all $k \in \{1, \ldots, n\}$ and remaining conditions of Theorem 2.8 hold, then*

$$\mathbb{E}\left((S_n^{\theta\xi})^\alpha \mathbb{1}_{\{S_n^{\theta\xi} > x\}}\right) \sim \sum_{k=1}^{n} \mathbb{E}\left((\theta_k \xi_k)^\alpha \mathbb{1}_{\{\theta_k \xi_k > x\}}\right).$$

Similar results corresponding to Remark 2.2 for unweighted sums were obtained as a particular case in [97].

**REMARK 2.3.** *The case $\alpha = 0$ corresponds precisely to Theorem 2.5.*

**REMARK 2.4.** *Let conditions of Theorem 2.8 hold. By combining* (2.6) *and* (2.8) *the following asymptotic bounds for conditional expectations could be obtained:*

$$\frac{\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\left((\theta_k \xi_k)^\alpha \mathbb{1}_{\{\theta_k \xi_k > x\}}\right)}{\sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \overline{F}_{\theta_k \xi_k}(x)} \lesssim \mathbb{E}\left((S_n^{\theta\xi})^\alpha | S_n^{\theta\xi} > x\right)$$

$$\lesssim \frac{\sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\left((\theta_k \xi_k)^\alpha \mathbb{1}_{\{\theta_k \xi_k > x\}}\right)}{\sum_{k=1}^{n} L_{F_{\xi_k}} \overline{F}_{\theta_k \xi_k}(x)}.$$

*In addition, by using the min-max inequality (see Lemma 2.1 in Subsection 2.3.1), the above asymtotic bounds could be expressed as:*

$$\min_{1 \leqslant k \leqslant n} \{L_{F_{\xi_k}}^2 \mathbb{E}((\theta_k \xi_k)^\alpha | \theta_k \xi_k > x)\} \lesssim \mathbb{E}\left((S_n^{\theta\xi})^\alpha | S_n^{\theta\xi} > x\right)$$

$$\lesssim \max_{1 \leqslant k \leqslant n} \{L_{F_{\xi_k}}^{-2} \mathbb{E}((\theta_k \xi_k)^\alpha | \theta_k \xi_k > x)\}.$$

## 2.2 Main Results

This section presents the main results of the chapter, which are published in [32] and are significantly influenced by [33]. The theorems below further generalize the previously discussed findings by permitting a broader class of transformations beyond degree functions. The first theorem in this section is formulated for sums of primary r.v.s and corresponds to Theorem 3 in [33], whereas the second theorem addresses randomly weighted sums and serves as an extension of Theorem 2.8.

**THEOREM 2.9.** *Let $\xi_1, \ldots, \xi_n$ be pQAI real-valued r.v.s such that $F_{\xi_k} \in \mathscr{D}$ for all $k \in \{1, \ldots, n\}$ and let $\varphi : \mathbb{R} \to \mathbb{R}_+ = [0, \infty)$ be a function which satisfies the following properties:*

(i) *$\varphi$ is nondecreasing for all sufficiently large $x$;*
(ii) *$\varphi$ is subhomogeneous for all sufficiently large $x$, i.e. $\varphi(2x) \leqslant c\varphi(x)$ for some positive constant $c$ and all sufficiently large $x$;*
(iii) *$\varphi$ is differentiable for all sufficiently large $x$;*
(iv) *for each $k \in \{1, \ldots, n\}$, the generalized moment $\mathbb{E}(\varphi(\xi_k)\mathbb{1}_{\{\xi_k > x\}})$ is finite for all sufficiently large $x$.*

*Then,*

$$\sum_{k=1}^{n} L_{\xi_k} \mathbb{E}\left(\varphi(\xi_k)\mathbb{1}_{\{\xi_k > x\}}\right) \lesssim \mathbb{E}\left(\varphi(S_n^{\xi})\mathbb{1}_{\{S_n^{\xi} > x\}}\right)$$

$$\lesssim \sum_{k=1}^{n} \frac{1}{L_{\xi_k}} \mathbb{E}\left(\varphi(\xi_k)\mathbb{1}_{\{\xi_k > x\}}\right).$$

**REMARK 2.5.** *It should be noted that requirements (i), (ii), (iii) impose conditions solely on the function $\varphi$, whereas condition (iv) involves not only function $\varphi$ but depends on the collection of r.v.s $\{\xi_1, \ldots, \xi_n\}$ as well. Examples of such functions can be found in Section 2.4.*

**THEOREM 2.10.** *Let $\xi_1, \ldots, \xi_n$ be pQAI real-valued r.v.s such that $F_{\xi_k} \in \mathscr{D}$ for all $k \in \{1, \ldots, n\}$ and let $\theta_1, \ldots, \theta_n$ be arbitrarily dependent, nonnegative, nondegenerate at zero r.v.s satisfying*

$$\max\{\mathbb{E}\theta_1^p, \ldots, \mathbb{E}\theta_n^p\} < \infty \quad \text{for some} \quad p > \max\{J^+_{F_{\xi_1}}, \ldots, J^+_{F_{\xi_n}}\}.$$

*Additionally, let the collections $\{\xi_1, \ldots, \xi_n\}$ and $\{\theta_1, \ldots, \theta_n\}$ be independent, and let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a function which satisfies properties* (i), (ii), (iii) *from Theorem 2.9 and the following one:*

(iv)* *the generalized moment $\mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{1}_{\{\theta_k\xi_k > x\}}\right)$ is finite for all $k \in \{1, \ldots, n\}$ and all sufficiently large $x$.*

*Then,*

$$\sum_{k=1}^{n} L_{\xi_k} \mathbb{E}\left(\varphi(\theta_k \xi_k) \mathbb{1}_{\{\theta_k \xi_k > x\}}\right) \lesssim \mathbb{E}\left(\varphi(S_n^{\theta\xi}) \mathbb{1}_{\{S_n^{\theta\xi} > x\}}\right)$$

$$\lesssim \sum_{k=1}^{n} \frac{1}{L_{\xi_k}} \mathbb{E}\left(\varphi(\theta_k \xi_k) \mathbb{1}_{\{\theta_k \xi_k > x\}}\right).$$

It is easily to verify that the function $\varphi(x) = x^{\alpha}$, $\alpha \geqslant 0$, satisfies properties (i), (ii) and (iii) in the theorems above. Consequently, the results of this section include those of [33] as special cases. Additionally, remarks analogous to those in 2.2, 2.3 and 2.4 continue to hold, since their conclusions follow from properties that are preserved in the more general setting addressed in this section.

## 2.3 Proofs of the Main Results

### 2.3.1 Auxiliary Lemmas

This subsection provides auxiliary statements that play a key role in the proofs of the main results. The first lemma, known as the min–max inequality, provides a simple double inequality for ratios of finite sums of real numbers.

**LEMMA 2.1.** *Let $l \in \mathbb{N}$, $a_i \geqslant 0$, $b_i > 0$, $i \in \{1, \ldots, l\}$. Then*

$$\min\left\{\frac{a_1}{b_1}, \ldots, \frac{a_l}{b_l}\right\} \leqslant \frac{a_1 + \ldots + a_l}{b_1 + \ldots + b_l} \leqslant \max\left\{\frac{a_1}{b_1}, \ldots, \frac{a_l}{b_l}\right\}.$$

*Proof.* The statement of the lemma follows directly once it is observed that

$$\frac{a_1 + \ldots + a_l}{b_1 + \ldots + b_l} = \sum_{i=1}^{l} \frac{a_i}{b_i} \frac{b_i}{b_1 + \ldots + b_l},$$

where $0 < \frac{b_i}{b_1 + \ldots + b_l} < 1$ for $i \in \{1, \ldots, l\}$. □

The following two statements relate to the alternative expectation formula presented as Theorem 1 in [74]. For additional information on alternative expectation formulas, see [83, 96] and the references therein.

**LEMMA 2.2.** *Let $\xi$ be a r.v. with the greatest lower bound $m$. Let $\varphi : [m, \infty) \to \mathbb{R}$ be a differentiable function with integrable derivative such that*

$$\int_m^\infty \Big( \int_m^x |\varphi'(t)| \mathrm{d}t \Big) \mathrm{d}F_\xi(x) < \infty.$$

*Then*

$$\mathbb{E}(\varphi(\xi)) = \varphi(m) + \int_m^\infty \varphi'(x) \overline{F}_\xi(x) \mathrm{d}x.$$

For the proof of the above result, see [74].

**LEMMA 2.3.** *Let $\xi$ be a real-valued r.v. supported on $\mathbb{R}$ and let $\varphi : [a, \infty) \to \mathbb{R}$ be a differentiable function such that*

$$\int_a^\infty \Big( \int_a^z |\varphi'(t)| \mathrm{d}t \Big) \mathrm{d}F_\xi(z) < \infty \tag{2.9}$$

*for some positive $a$. Then,*

$$\mathbb{E}(\varphi(\xi) \mathbb{1}_{\{\xi > x\}}) = \varphi(x) \overline{F}_\xi(x) + \int_x^\infty \varphi'(t) \overline{F}_\xi(t) \mathrm{d}t \tag{2.10}$$

*for all $x \geqslant a$.*

*Proof.* The statement of the lemma can be derived from Lemma 2.2 or can be proved directly using Fubini's theorem. The latter option is chosen in this proof. For a given $x \geqslant a$, it holds that

$$\mathbb{E}\left( (\varphi(\xi) - \varphi(x)) \mathbb{1}_{\{\xi > x\}} \right) = \int_{-\infty}^\infty (\varphi(u) - \varphi(x)) \mathbb{1}_{\{u > x\}} \mathrm{d}F_\xi(u)$$

$$= \int_x^\infty (\varphi(u) - \varphi(x)) \mathrm{d}F_\xi(u) = \int_x^\infty \Big( \int_x^u \varphi'(t) \mathrm{d}t \Big) \mathrm{d}F_\xi(u).$$

Under condition (2.9), the Fubini-Tonelli theorem applies, and therefore

$$\int_x^\infty \Big( \int_x^u \varphi'(t) \mathrm{d}t \Big) \mathrm{d}F_\xi(u) = \int_x^\infty \Big( \int_t^\infty \varphi'(t) \mathrm{d}F_\xi(u) \Big) \mathrm{d}t$$

$$= \int_x^\infty \varphi'(t)\Big(\int_t^\infty \mathrm{d}F_\xi(u)\Big)\mathrm{d}t = \int_x^\infty \varphi'(t)(1 - F_\xi(t))\mathrm{d}t = \int_x^\infty \varphi'(t)\overline{F}_\xi(t)\mathrm{d}t.$$

The two derived formulas together yield equality (2.10), which completes the proof of Lemma 2.3. □

The following lemma is the result of [68] and corresponds to Remark 2.3. For completeness, the asymptotic behaviour of tail probabilities of unweighted sums is presented below, as it is crucial for the proof of Theorem 2.9.

**LEMMA 2.4.** *Let $\xi_1, \ldots, \xi_n$ be pQAI real-valued r.v.s, such that $F_{\xi_k} \in \mathscr{D}$ for all $k \in \{1, \ldots, n\}$. Then*

$$\sum_{k=1}^n L_{\xi_k}\overline{F}_{\xi_k}(x) \lesssim \mathbb{P}(S_n^\xi > x) \lesssim \sum_{k=1}^n \frac{1}{L_{\xi_k}}\overline{F}_{\xi_k}(x).$$

The proof of the lemma above can be found in [33].

For the proof of Theorem 2.10, two additional lemmas are required. The proof of Lemma 2.5 is given in [58] (see Lemma 3), while the proof of Lemma 2.6 is provided in [33] (see Lemma 4).

**LEMMA 2.5.** *Let $\xi$ and $\theta$ be two independent r.v.s such that $F_\xi \in \mathscr{D}$ and $\theta$ is nonnegative, nondegenerate at zero r.v, then $F_{\theta\xi} \in \mathscr{D}$, where $F_{\theta\xi}(x) := \mathbb{P}(\theta\xi \leqslant x)$.*

*If, in addition,*

$$\mathbb{E}\theta^p < \infty \quad \textit{for some} \quad p > J_\xi^+ = \inf_{y>1}\left\{-\frac{1}{\log y}\log\left\{\liminf_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}\right\}\right\},$$

*then the inequality $L_{\theta\xi} \geqslant L_\xi$ holds for L-indices.*

**LEMMA 2.6.** *Let $\xi_1$, $\xi_2$ be QAI r.v.s such that $F_{\xi_k} \in \mathscr{D}$, $k \in \{1, 2\}$, and let $\theta_1$, $\theta_2$ be two arbitrarily dependent, nonnegative, nondegenerate at zero r.v.s with $\max\{\mathbb{E}\theta_1^p, \mathbb{E}\theta_2^p\} < \infty$ for some $p > \max\{J_{\xi_1}^+, J_{\xi_2}^+\}$. Then r.v.s $\theta_1\xi_1$ and $\theta_2\xi_2$ are QAI.*

### 2.3.2 Proof of Theorem 2.9

The case $n = 1$ follows trivially from the definition of coefficient $L_{F_{\xi_1}}$. Let $n \geqslant 2$. Since $n$ is fixed, the conditions of the theorem imply that requirements (i)-(iv) are satisfied for all $x \geqslant N$, when $N$ is sufficiently large.

Due to properties (i) and (ii), one can obtain $\varphi(nx) = O(\varphi(x))$. Indeed, since $\varphi(2x) \leqslant c(\varphi(x))$ for $x \geqslant N$, the following inequalities hold:

$$\varphi(nx) = \varphi\left(2\frac{nx}{2}\right) \leqslant c\varphi\left(\frac{nx}{2}\right) \text{ for } \frac{nx}{2} \geqslant N,$$

$$c\varphi\left(\frac{nx}{2}\right) = c\varphi\left(2\frac{nx}{4}\right) \leqslant c^2\varphi\left(\frac{nx}{4}\right) \text{ for } \frac{nx}{4} \geqslant N,$$

$$\dots,$$

$$c^{m-1}\varphi\left(\frac{nx}{2^{m-1}}\right) = c^{m-1}\varphi\left(2\frac{nx}{2^m}\right) \leqslant c^m\varphi\left(\frac{nx}{2^m}\right) \text{ for } \frac{nx}{2^m} \geqslant N.$$

Therefore, for each given $m \in \mathbb{N}$ and $x \geqslant 2^m N/n$:

$$\varphi(nx) \leqslant c^m\varphi\left(\frac{nx}{2^m}\right).$$

Choosing $m$ such that $\frac{n}{2^m} \leqslant 1$ and using the fact that $\varphi$ is nondecreasing for sufficiently large $x$, one can get that there is $K \geqslant N$ such that

$$\varphi(nx) \leqslant c^m\varphi\left(\frac{nx}{2^m}\right) \leqslant c^m\varphi(x) \leqslant c^{\lfloor \log_2 n \rfloor + 1}\varphi(x)$$

for all $x \geqslant K$. Therefore property (iv), together with the derived estimate, imples that

$$\mathbb{E}\left(\varphi(S_n^\xi)\mathbb{1}_{\{S_n^\xi > \widehat{M}\}}\right) < \infty$$

for some $\widehat{M} > 0$.

Indeed, there exists $\widehat{K} \geqslant K$ such that $0 < \varphi(x_1) \leqslant \varphi(x_2)$ for $\widehat{K} < x_1 < x_2$ and

$$\mathbb{E}\left(\varphi(\xi_k)\mathbb{1}_{\{\xi_k > \widehat{K}\}}\right) < \infty$$

for $k \in \{1, \dots, n\}$. Thus, for $\widehat{M} = n\widehat{K}$, one can get

$$\mathbb{E}\left(\varphi(S_n^\xi)\mathbb{1}_{\{S_n^\xi > \widehat{M}\}}\right) \leqslant \mathbb{E}\left(\varphi(n\max\{\xi_1, \dots, \xi_n\})\mathbb{1}_{\{\max\{\xi_1,\dots,\xi_n\} > \widehat{K}\}}\right)$$

$$\leqslant 2^{\log_2 n + 1}\sum_{k=1}^n \mathbb{E}\left(\varphi(n\xi_k)\mathbb{1}_{\{\xi_k > \widehat{K}\}}\right)$$

$$\leqslant 2^{\log_2 n + 1}\sum_{k=1}^n \mathbb{E}\left(\varphi(\xi_k)\mathbb{1}_{\{\xi_k > \widehat{K}\}}\right) < \infty.$$

Hence, due to properties (i) and (iii), there exists $M > \widehat{M}$ such that

$$\int_M^\infty \left( \int_M^u |\varphi'(t)| \mathrm{d}t \right) \mathrm{d}F_{S_n^\xi}(u) = \int_M^\infty \left( \int_M^u \varphi'(t) \mathrm{d}t \right) \mathrm{d}F_{S_n^\xi}(u)$$
$$= \mathbb{E}\left( \varphi(S_n^\xi) \mathbb{1}_{\{S_n^\xi \geqslant M\}} - \varphi(M) \right) < \infty.$$

It follows that Lemma 2.3 is applicable for the truncated sum $S_n^\xi \mathbb{1}_{\{S_n^\xi > x\}}$. Using Lemmas 2.1 and 2.3, for $x$ large enough ($x \geqslant M$),

$$\frac{\mathbb{E}\left( \varphi(S_n^\xi) \mathbb{1}_{\{S_n^\xi > x\}} \right)}{\sum_{k=1}^n \frac{1}{L_{\xi_k}} \mathbb{E}\left( \varphi(\xi_k) \mathbb{1}_{\{\xi_k > x\}} \right)}$$

$$= \frac{\varphi(x) \mathbb{P}\left( S_n^\xi > x \right) + \int_x^\infty \varphi'(u) \mathbb{P}\left( S_n^\xi > u \right) \mathrm{d}u}{\sum_{k=1}^n \frac{1}{L_{\xi_k}} \left( \varphi(x) \overline{F}_{\xi_k}(x) + \int_x^\infty \varphi'(u) \overline{F}_{\xi_k}(u) \mathrm{d}u \right)}$$

$$\leqslant \max \left\{ \frac{\mathbb{P}\left( S_n^\xi > x \right)}{\sum_{k=1}^n \frac{1}{L_{\xi_k}} \overline{F}_{\xi_k}(x)}, \frac{\int_x^\infty \varphi'(u) \mathbb{P}\left( S_n^\xi > u \right) \mathrm{d}u}{\int_x^\infty \varphi'(u) \sum_{k=1}^n \frac{1}{L_{\xi_k}} \overline{F}_{\xi_k}(u) \mathrm{d}u} \right\}$$
$$=: \max \left\{ \mathcal{C}_1(x), \mathcal{C}_2(x) \right\}.$$

By Lemma 2.4, one can obtain

$$\limsup_{x \to \infty} \mathcal{C}_1(x) \leqslant 1,$$

and for the term $\mathcal{C}_2(x)$, one can obtain

$$\limsup_{x \to \infty} \mathcal{C}_2(x) = \limsup_{x \to \infty} \frac{\int_x^\infty \varphi'(u) \mathbb{P}\left( S_n^\xi > u \right) \mathrm{d}u}{\int_x^\infty \varphi'(u) \mathbb{P}\left( S_n^\xi > u \right) \frac{\sum_{k=1}^n \frac{1}{L_{\xi_k}} \overline{F}_{\xi_k}(u)}{\mathbb{P}\left( S_n^\xi > u \right)} \mathrm{d}u}$$
$$\leqslant \limsup_{x \to \infty} \sup_{u \geqslant x} \frac{\mathbb{P}\left( S_n^\xi > u \right)}{\sum_{k=1}^n \frac{1}{L_{\xi_k}} \overline{F}_{\xi_k}(u)} \leqslant \limsup_{x \to \infty} \frac{\mathbb{P}\left( S_n^\xi > x \right)}{\sum_{k=1}^n \frac{1}{L_{\xi_k}} \overline{F}_{\xi_k}(x)} \leqslant 1,$$

where last inequality follows from Lemma 2.4 as well. This proves the desired upper estimate in Theorem 2.9.

The asymptotic lower bound in Theorem 2.9 follows similarly. Indeed, in the same fashion, one can obtain

$$\frac{\mathbb{E}\left(\varphi(S_n^\xi)\mathbb{1}_{\{S_n^\xi>x\}}\right)}{\sum\limits_{k=1}^{n} L_{\xi_k}\mathbb{E}\left(\varphi(\xi_k)\mathbb{1}_{\{\xi_k>x\}}\right)}$$

$$=\frac{\varphi(x)\mathbb{P}\left(S_n^\xi>x\right)+\int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi>u\right)\mathrm{d}u}{\sum\limits_{k=1}^{n} L_{\xi_k}\left(\varphi(x)\overline{F}_{\xi_k}(x)+\int\limits_x^\infty \varphi'(u)\overline{F}_{\xi_k}(u)\mathrm{d}u\right)}$$

$$\geqslant \min\left\{\frac{\mathbb{P}\left(S_n^\xi>x\right)}{\sum\limits_{k=1}^{n} L_{\xi_k}\overline{F}_{\xi_k}(x)},\frac{\int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi>u\right)\mathrm{d}u}{\int\limits_x^\infty \varphi'(u)\sum\limits_{k=1}^{n} L_{\xi_k}\overline{F}_{\xi_k}(u)\mathrm{d}u}\right\}$$

$$=:\min\left\{\mathcal{C}_3(x),\mathcal{C}_4(x)\right\}.$$

By Lemma 2.4, one can obtain

$$\liminf_{x\to\infty}\mathcal{C}_3(x)\geqslant 1,$$

and, for the term $\mathcal{C}_4(x)$, one can obtain

$$\liminf_{x\to\infty}\mathcal{C}_4(x)=\liminf_{x\to\infty}\frac{\int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi>u\right)\mathrm{d}u}{\int\limits_x^\infty \varphi'(u)\mathbb{P}\left(S_n^\xi>u\right)\frac{\sum_{k=1}^{n} L_{\xi_k}\overline{F}_{\xi_k}(u)}{\mathbb{P}\left(S_n^\xi>u\right)}\mathrm{d}u}$$

$$\geqslant\liminf_{x\to\infty}\inf_{u\geqslant x}\frac{\mathbb{P}\left(S_n^\xi>u\right)}{\sum\limits_{k=1}^{n} L_{\xi_k}\overline{F}_{\xi_k}(u)}=\lim_{x\to\infty}\inf_{u\geqslant x}\frac{\mathbb{P}\left(S_n^\xi>u\right)}{\sum\limits_{k=1}^{n} L_{\xi_k}\overline{F}_{\xi_k}(u)}$$

$$=\liminf_{x\to\infty}\frac{\mathbb{P}\left(S_n^\xi>x\right)}{\sum\limits_{k=1}^{n} L_{\xi_k}\overline{F}_{\xi_k}(x)}\geqslant 1,$$

where last inequality follows from Lemma 2.4. This proves the desired lower estimate in Theorem 2.9.

Theorem 2.9 is proved. $\square$

### 2.3.3 Proof of Theorem 2.10

As $\max\{\mathbb{E}\theta_1^p, \ldots, \mathbb{E}\theta_n^p\} < \infty$ for some $p > \max\{J_{\xi_1}^+, \ldots, J_{\xi_n}^+\}$, Lemma 2.5 implies that $F_{\theta_k \xi_k} \in \mathscr{D}$ and $L_{\theta_k \xi_k} \geqslant L_{\xi_k}$, for all $k \in \{1, \ldots, n\}$. Moreover, according to Lemma 2.6, for any $k, l \in \{1, \ldots, n\}$, $k \neq l$, r.v.s $\theta_k \xi_k$ and $\theta_l \xi_l$ are QAI. In other words, r.v.s $\theta_1 \xi_1, \ldots, \theta_n \xi_n$ are pQAI. Finally, one can apply Theorem 2.9 to r.v.s. $\theta_1 \xi_1, \ldots, \theta_n \xi_n$ and obtain the desired result. $\square$

## 2.4 Examples

This section presents specific examples illustrating the asymptotic relations stated in Theorems 2.9 and 2.10. For simplicity, sums $S_n^{\xi}$ and $S_n^{\theta\xi}$ are restricted to cases with two or three summands. In all examples, r.v.s $\xi_1, \xi_2, \xi_3$ are assumed to satisfy the pQAI condition and are constructed via a two-dimensional or three-dimensional FGM copula (see Example 1.7 for details). Moreover, collections $\{\theta_1, \theta_2, \theta_3\}$ and $\{\xi_1, \xi_2, \xi_3\}$ are considered to be independent.

Since analytic expressions for tail expectations $\mathbb{E}(\varphi(S_n^{\xi}) \mathbb{1}_{\{S_n^{\xi} > x\}})$ and $\mathbb{E}(\varphi(S_n^{\theta\xi}) \mathbb{1}_{\{S_n^{\theta\xi} > x\}})$ may be difficult or impossible to derive, these quantities are estimated using the Monte-Carlo simulation method. Likewise, Monte-Carlo estimates are provided for sums of tail expectations of individual summands $\mathbb{E}(\varphi(\xi_k) \mathbb{1}_{\{\xi_k > x\}})$ and $\mathbb{E}(\varphi(\theta_k \xi_k) \mathbb{1}_{\{\theta_k \xi_k > x\}})$. Details of the sampling procedure may be found in Section 6.1 of [33]. In addition, tail expectations of individual summands are evaluated by the asymptotically equivalent expressions.

**EXAMPLE 2.1.** *Consider three Pareto distributions (see Example 1.6) $\xi_1 \overset{d}{=} \mathcal{P}a_{\{2,1\}}$, $\xi_2 \overset{d}{=} \mathcal{P}a_{\{2,2\}}$, $\xi_3 \overset{d}{=} \mathcal{P}a_{\{2,3\}}$ which are pQAI. More specifically, suppose that random vector $(\xi_1, \xi_2, \xi_3)$ follows the three-dimensional FGM copula*

$$\mathbb{P}(\xi_1 \leqslant x_1, \xi_2 \leqslant x_2, \xi_3 \leqslant x_3) = F_{\xi_1}(x_1) F_{\xi_2}(x_2) F_{\xi_3}(x_3)$$
$$\times \Big(1 + \frac{1}{3} \overline{F}_{\xi_1}(x_1) \overline{F}_{\xi_2}(x_2) + \frac{1}{3} \overline{F}_{\xi_2}(x_2) \overline{F}_{\xi_3}(x_3) + \frac{1}{3} \overline{F}_{\xi_1}(x_1) \overline{F}_{\xi_3}(x_3)\Big)$$

*In addition, consider the generalized moment function* $\varphi(x) = \frac{x}{\log x}$. *In this example, the asymptotic formula is derived for*

$$\mathbb{E}\left(\varphi(S_3^\xi)\,\mathbb{1}_{\{S_3^\xi > x\}}\right) = \mathbb{E}\left(\left(\frac{\xi_1+\xi_2+\xi_3}{\log(\xi_1+\xi_2+\xi_3)}\right)\mathbb{1}_{\{\xi_1+\xi_2+\xi_3>x\}}\right).$$

*It is easy to verify that all conditions of Theorem 2.9 are satisfied. Moreover,* $L_{\xi_1} = L_{\xi_2} = L_{\xi_3} = 1$. *Hence,*

$$\mathbb{E}\left(\varphi(S_3^\xi)\,\mathbb{1}_{\{S_3^\xi > x\}}\right) \sim \sum_{k=1}^{3}\mathbb{E}\left(\frac{\xi_k}{\log \xi_k}\mathbb{1}_{\{\xi_k>x\}}\right) \sim \frac{28}{x\log x}$$

*because*

$$\mathbb{E}\left(\frac{\xi_k}{\log \xi_k}\mathbb{1}_{\{\xi_k>x\}}\right) = 2k^2\int\limits_x^\infty \frac{y}{\log y}\frac{\mathrm{d}y}{(k+y)^3} \sim 2k^2\int\limits_x^\infty \frac{\mathrm{d}y}{y^2\log y}$$

$$= 2k^2\int\limits_{\log x}^\infty \frac{e^{-z}}{z}\mathrm{d}z \sim \frac{2k^2}{x\log x},$$

*for all* $k \in \{1,2,3\}$.

*The results of simulated values together with the derived analytical expression are presented in Figure 2.1.*

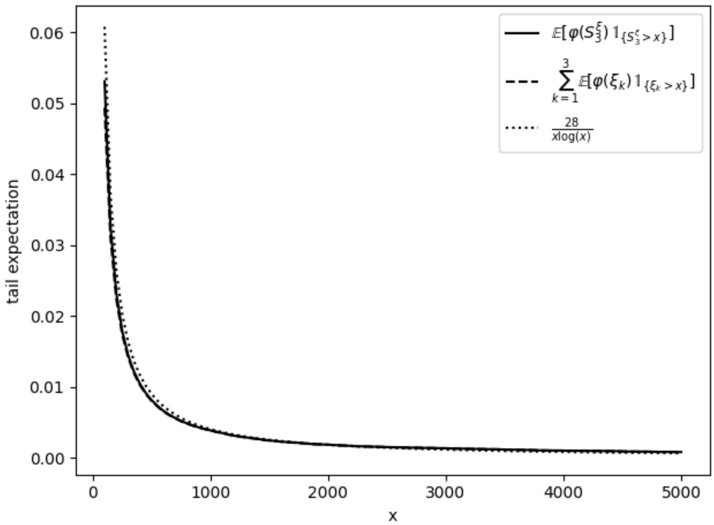


**FIGURE 2.1.** Asymptotic relations for the truncated generalized moment of Example 2.1.

**EXAMPLE 2.2.** *Let the assumptions of Example 2.1 continue to hold. Furthermore, consider independent random weights $\theta_1, \theta_2, \theta_3$ distributed according to the Bernoulli distribution shifted by 1 with $p_1 = \frac{1}{10}$, $p_2 = \frac{1}{4}$, and $p_3 = \frac{1}{2}$, respectively, meaning that, for example, $\mathbb{P}(\theta_1 = 2) = \frac{1}{10}, \mathbb{P}(\theta_1 = 1) = \frac{9}{10}$. It is obvious that all conditions of Theorem 2.10 are satisfied. Moreover, it can be demonstrated that*

$$\mathbb{E}\left(\frac{\theta_k \xi_k}{\log \theta_k \xi_k} \mathbb{1}_{\{\theta_k \xi_k > x\}}\right) \sim \frac{p_k 8k^2 + (1-p_k)2k^2}{x \log x}$$

*for all $k \in \{1,2,3\}$.*

*Figure 2.2 shows results of the following asymptotic relations*

$$\mathbb{E}\left(\frac{S_3^{\theta\xi}}{\log S_3^{\theta\xi}} \mathbb{1}_{\{S_3^{\theta\xi} > x\}}\right) \sim \sum_{k=1}^{3} \mathbb{E}\left(\frac{\theta_k \xi_k}{\log \theta_k \xi_k} \mathbb{1}_{\{\theta_k \xi_k > x\}}\right) \sim \frac{61.6}{x \log x}.$$

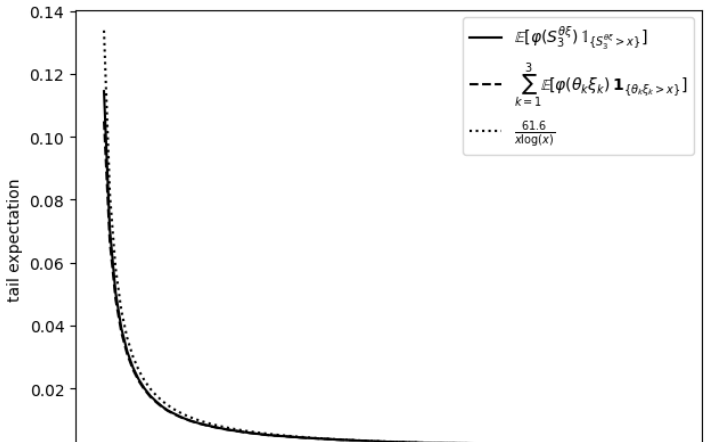


**FIGURE 2.2.** Asymptotic relations for the truncated generalized moment of Example 2.2.

**EXAMPLE 2.3.** *Consider two r.v.s distributed according to Peter and Paul laws $\xi_1 \overset{d}{=} \mathcal{PP}_{\{1,2\}}$, $\xi_2 \overset{d}{=} \mathcal{PP}_{\{1,\sqrt{2}\}}$ (see Example 1.4) which are pQAI. More specifically, suppose that random vector $(\xi_1, \xi_2)$ follows the two-dimensional FGM copula*

$$\mathbb{P}(\xi_1 \leqslant x_1, \xi_2 \leqslant x_2) = F_{\xi_1}(x_1) F_{\xi_2}(x_2)\left(1 + \frac{1}{2}\overline{F}_{\xi_1}(x_1)\overline{F}_{\xi_2}(x_2)\right).$$

*In addition, consider the generalized moment function $\varphi(x) = \sqrt[3]{x}\log x$. In this example, the asymptotic formula is derived for*

$$\Delta(x) := \mathbb{E}\left(\varphi\big(S_2^{\xi}\big)\mathbb{1}_{\{S_2^{\xi}>x\}}\right) = \mathbb{E}\left(\left(\sqrt[3]{\xi_1+\xi_2}\log(\xi_1+\xi_2)\right)\mathbb{1}_{\{\xi_1+\xi_2>x\}}\right).$$

*Since all of the conditions of Theorem 2.9 are satisfied, one can obtain*

$$\sum_{k=1}^{2} L_{\xi_k}\mathbb{E}(\sqrt[3]{\xi_k}\log(\xi_k)\mathbb{1}_{\{\xi_k>x\}}) \lesssim \Delta(x) \lesssim \sum_{k=1}^{2}\frac{1}{L_{\xi_k}}\mathbb{E}(\sqrt[3]{\xi_k}\log(\xi_k)\mathbb{1}_{\{\xi_k>x\}}).$$

*According to the definition of $\mathcal{PP}$ distribution,*

$$\mathbb{P}(\xi_1 = 2^k) = 2^{-k}, \quad \mathbb{P}(\xi_2 = 2^{\frac{k}{2}}) = (\sqrt{2}-1)2^{-\frac{k}{2}}, \quad k \in \{1, 2, \ldots\},$$

*and*

$$\overline{F}_{\xi_1}(x) = 2^{-\lfloor \log_2 x\rfloor}, \quad \overline{F}_{\xi_2}(x) = 2^{-\frac{\lfloor \log_{\sqrt{2}} x\rfloor}{2}}, \quad x \geqslant 1.$$

*The latter expressions imply that $L_{\xi_1} = \frac{1}{2}$ and $L_{\xi_2} = \frac{1}{\sqrt{2}}$ (see [33] for details). In addition,*

$$\mathbb{E}(\sqrt[3]{\xi_1}\log(\xi_1)\mathbb{1}_{\{\xi_1>x\}}) = \log 2 \sum_{k=\lfloor \log_2 x\rfloor+1}^{\infty} k2^{-\frac{2k}{3}}$$
$$= \frac{\sqrt[3]{2}\log 2}{(2-\sqrt[3]{2})^2}2^{-\frac{2\lfloor \log_2 x\rfloor}{3}}(2+(2-\sqrt[3]{2})\lfloor \log_2 x\rfloor),$$

*and, similarly,*

$$\mathbb{E}(\sqrt[3]{\xi_2}\log(\xi_2)\mathbb{1}_{\{\xi_2>x\}}) = \frac{(\sqrt{2}-1)\log 2}{2}\sum_{k=\lfloor 2\log_2 x\rfloor+1}^{\infty} k2^{-\frac{k}{3}}$$
$$= \frac{(\sqrt{2}-1))\sqrt[3]{2}\log 2}{(2-\sqrt[3]{4})^2}2^{-\frac{\lfloor 2\log_2 x\rfloor}{3}}(\sqrt[3]{2}+(\sqrt[3]{2}-1)\lfloor 2\log_2 x\rfloor).$$

*The last relations and Theorem 2.9 imply that*

$$\Delta(x) \lesssim \sqrt[3]{2}\log 2\Bigg(\frac{2}{(2-\sqrt[3]{2})^2}2^{-\frac{2\lfloor \log_2 x\rfloor}{3}}(2+(2-\sqrt[3]{2})\lfloor \log_2 x\rfloor)$$
$$+\frac{2-\sqrt{2}}{(2-\sqrt[3]{4})^2}2^{-\frac{\lfloor 2\log_2 x\rfloor}{3}}(\sqrt[3]{2}+(\sqrt[3]{2}-1)\lfloor 2\log_2 x\rfloor)\Bigg)$$

$$\Delta(x) \gtrsim \sqrt[3]{2}\log 2\Bigg(\frac{1}{2(2-\sqrt[3]{2})^2}2^{-\frac{2\lfloor\log_2 x\rfloor}{3}}(2+(2-\sqrt[3]{2})\lfloor\log_2 x\rfloor)$$
$$+\frac{\sqrt{2}-1}{\sqrt{2}(2-\sqrt[3]{4})^2}2^{-\frac{\lfloor 2\log_2 x\rfloor}{3}}(\sqrt[3]{2}+(\sqrt[3]{2}-1)\lfloor 2\log_2 x\rfloor)\Bigg).$$

*The results of simulated values together with the derived analytical bounds are presented in Figure 2.3.*

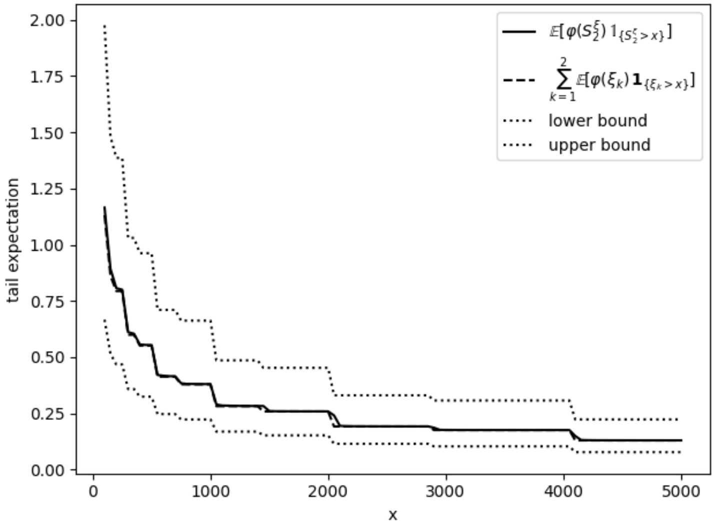


**FIGURE 2.3.** Asymptotic relations for the truncated generalized moment of Example 2.3.

# 3 REGULARITY OF THE STOPPING MOMENT

In the previous chapter, sums with a fixed number of summands were examined. This chapter turns to randomly stopped sums, where primary r.v.s $\xi_1, \xi_2, \ldots$ are assumed to be nonnegative, independent, and identically distributed (i.i.d.), and the stopping moment is a nonnegative, nondegenerate at zero, integer-valued r.v., independent of primary r.v.s.

The associated sequence of partial sums $\{S_n, n \geqslant 0\}$ is defined by

$$S_0 := 0, \quad S_n := \xi_1 + \ldots + \xi_n, \quad n \geqslant 1. \tag{3.1}$$

The main subject of this chapter is the study of randomly stopped sum

$$S_\nu := \begin{cases} 0 & \text{if } \nu = 0, \\ \xi_1 + \ldots + \xi_\nu & \text{if } \nu \geqslant 1, \end{cases}$$

where $n$ in (3.1) is replaced by counting r.v. $\nu$ taking values in $\mathbb{N}_0$. Here, $\{\xi_1, \xi_2, \ldots\}$ is a sequence of independent copies of a r.v. $\xi$ with d.f. $F_\xi(x)$. In this setting, the d.f. of the randomly stopped sum is

$$F_{S_\nu}(x) = \mathbb{P}(S_\nu \leqslant x) = \sum_{n=0}^{\infty} \mathbb{P}(S_n \leqslant x)\mathbb{P}(\nu = n),$$

and, for positive $x$, the corresponding tail function has the form

$$\overline{F}_{S_\nu}(x) = \mathbb{P}(S_\nu > x) = \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\nu = n).$$

The theory of randomly stopped sums for various subclasses of heavy-tailed r.v.s has been extensively developed over the past fifty years, primarily within the framework of i.i.d. summands. This chapter continues to consider this standard model. Research of randomly stopped sums plays an important role in insurance and financial mathematics, survival analysis, risk theory, computer and communication networks (see, e.g., [8, 13, 18, 21, 34, 39, 46, 61, 62, 94, 100, 102, 108, 119, 128]).

The rest of this chapter is structured as follows. Section 3.1 provides a review of known results on randomly stopped sums. Section 3.2 presents new results identifying conditions under which the regularity of a randomly stopped sum determines regularity of the stopping moment. Proofs of these results are given in Section 3.3, followed by an illustrative example in Section 3.4.

## 3.1 Background

This section begins with a brief review of direct type results on the regularity of randomly stopped sums, that is, conditions are specified for random summands and counting r.v. ensuring that randomly stopped sums fall into the desired regularity class.

Randomly stopped sums consisting of regularly varying r.v.s were considered in [37, 39, 89, 98, 99]. For example, Faÿ et al. (see Proposition 4.1 in [39]) achieved the following:

**THEOREM 3.1.** *Let $\xi$ be a nonnegative r.v. with d.f. $F_\xi \in \mathscr{R}_\alpha$, $\alpha > 0$. Additionally, assume $\mathbb{E}\nu < \infty$ and $\overline{F}_\nu(x) = o(\overline{F}_\xi(x))$. Then*

$$\overline{F}_{S_\nu}(x) \sim \mathbb{E}\nu \overline{F}_\xi(x).$$

Problems involving randomly stopped sums composed of consistently varying r.v.s have been examined in [3, 18, 22, 60]. For example, Chen and Yuen (see Theorem 3.4 in [22]) obtained results analogous to those presented in Theorem 3.1:

**THEOREM 3.2.** *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of pQAI real-valued r.v.s with common d.f. $F_\xi \in \mathscr{C}$, and let $\nu$ be a nonnegative and integer-valued r.v. independent of the sequence $\{\xi_1, \xi_2, \ldots\}$. If $0 < \mathbb{E}\nu^{q+1} < \infty$ for some $q > J^+_{F_\xi}$, then*

$$\overline{F}_{S_\nu}(x) \sim \mathbb{E}\nu \overline{F}_\xi(x).$$

The asymptotic properties of randomly stopped sums with dominatedly varying summands were examined in [30, 69, 70, 101, 129]. For instance, Leipus and Šiaulys (see Theorem 5 in [69]) proved the following:

**THEOREM 3.3.** *Let $\xi_1, \xi_2, \ldots$ be i.i.d. nonnegative r.v.s with common d.f. $F_\xi \in \mathscr{D}$ and finite mean $\mathbb{E}\xi$. Additionally, let $\nu$ be a nonnegative,* *integer-valued r.v. independent of* $\xi_1, \xi_2, \ldots$*, with d.f.* $F_\nu$ *and finite mean* $\mathbb{E}\nu > 0$*. Then*

$$F_{S_\nu} \in \mathscr{D} \quad \Longleftrightarrow \quad \min\{F_\xi, F_\nu\} \in \mathscr{D}.$$

Problems involving randomly stopped sums consisting of long-tailed r.v.s have been considered in [26, 31, 36, 69, 70, 119, 121, 122]. Note that asymptotic properties in these papers are found either exclusively for summands with long tails or for summands satisfying additional conditions. For example, Denisov et al. (see Theorem 8 in [31]) stated the following:

**THEOREM 3.4.** *Let* $\xi_1, \xi_2, \ldots$ *be i.i.d. r.v.s with common d.f.* $F_\xi \in \mathscr{L} \cap \mathscr{D}$ *and* $\mathbb{E}\xi > 0$*. Additionally, let* $\nu$ *be independent of* $\xi_1, \xi_2, \ldots$*, with d.f.* $F_\nu \in \mathscr{C}$*. Then*

$$\overline{F}_{S_\nu}(x) \sim \mathbb{E}\nu \overline{F}_\xi(x) + \overline{F}_\nu\left(\frac{x}{\mathbb{E}\xi}\right).$$

The next part of this section provides two inverse type results (see Proposition 4.9 in [39]) that are closely related to the main findings of this chapter. In these cases, the regularity of randomly stopped sums is specified.

**THEOREM 3.5.** *Let* $\{\xi_1, \xi_2, \ldots\}$ *be a sequence of i.i.d copies of a non-negative r.v.* $\xi$*, and let* $\nu$ *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$*. If the d.f. of the randomly stopped sum* $F_{S_\nu} \in \mathscr{R}_\alpha$ *with* $\alpha > 0$*,* $\mathbb{E}\xi < \infty$*,* $\mathbb{E}\nu < \infty$*, and* $\overline{F}_\xi(x) = o(\overline{F}_{S_\nu}(x))$*, then* $F_\nu \in \mathscr{R}_\alpha$*, and*

$$\overline{F}_{S_\nu}(x) \sim (\mathbb{E}\xi)^\alpha \overline{F}_\nu(x).$$

**THEOREM 3.6.** *Let the sequence of r.v.s* $\{\xi_1, \xi_2, \ldots\}$ *and r.v. satisfy the basic conditions of Theorem 3.5. If* $F_{S_\nu} \in \mathscr{R}_1$*,* $\mathbb{E}\xi < \infty$*,* $\mathbb{E}\nu = \infty$*, and* $x\overline{F}_\xi(x) = o(\overline{F}_{S_\nu}(x))$*, then* $F_\nu \in \mathscr{R}_1$*, and*

$$\overline{F}_{S_\nu}(x) \sim \mathbb{E}\xi \overline{F}_\nu(x).$$

Similar inverse problems for other transformations of distributions are considered in [29, 35, 63, 84, 88, 120, 122, 124, 125].

This section extends the results of Theorems 3.5 and 3.6 to broader subclasses of heavy-tailed distributions, namely $\mathscr{ERV}$ and $\mathscr{D}$. In particular, it is shown how the regularity of a randomly stopped sum - whether it belongs to $\mathscr{ERV}$ or $\mathscr{D}$ - determines the regularity of the counting r.v. in the class $\mathscr{D}$. Moreover, when the distribution of a randomly stopped sum belongs to the class $\mathscr{ERV}$, asymptotic expressions analogous to those in Theorems 3.5 and 3.6 continue to hold. All of the findings discussed in this section are based on the publication [91].

**THEOREM 3.7.** *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of i.i.d copies of a non-negative r.v. $\xi$, and let $\nu$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If the d.f. of the randomly stopped sum $F_{S_\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$ with $0 \leqslant \alpha \leqslant \beta < \infty$, $\mathbb{E}\xi < \infty$, $\mathbb{E}\nu < \infty$, and $\overline{F}_\xi(x) = o(\overline{F}_{S_\nu}(x))$, then $F_\nu \in \mathscr{D}$, and*

$$(\mathbb{E}\xi)^\alpha \overline{F}_\nu(x) \lesssim \overline{F}_{S_\nu}(x) \lesssim (\mathbb{E}\xi)^\beta \overline{F}_\nu(x) \quad \text{if} \quad \mathbb{E}\xi > 1 \tag{3.2}$$

$$(\mathbb{E}\xi)^\beta \overline{F}_\nu(x) \lesssim \overline{F}_{S_\nu}(x) \lesssim (\mathbb{E}\xi)^\alpha \overline{F}_\nu(x) \quad \text{if} \quad \mathbb{E}\xi < 1 \tag{3.3}$$

$$\overline{F}_{S_\nu}(x) \sim \overline{F}_\nu(x) \quad \text{if} \quad \mathbb{E}\xi = 1 \tag{3.4}$$

It is evident that Theorem 3.7 implies Theorem 3.5 when $\alpha = \beta$.

**REMARK 3.1.** *The conditions of Theorem 3.7 imply that the expectation of the primary r.v. $\xi$ is positive, i.e., $\mathbb{E}\xi > 0$.*

*Indeed, assuming to the contrary that $\mathbb{E}\xi = 0$, it follows that $\mathbb{P}(\xi = 0) = 1$. In this case, $\mathbb{P}(S_n = 0) = 1$ for all $n$. Hence,*

$$\overline{F}_{S_\nu}(x) = \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\nu = n) = 0$$

*for $x > 0$. This contradicts to the condition $F_{S_\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$, therefore $\mathbb{P}(\xi > 0) > 0$, implying $\mathbb{E}\xi > 0$.*

**REMARK 3.2.** *The conditions $F_{S_\nu} \in \mathscr{ERV} \subset \mathscr{D}$ and $\overline{F}_\xi(x) = o(\overline{F}_{S_\nu}(x))$ imply that the support of the counting r.v. $\nu$ is infinite, i.e., $\overline{F}_\nu(x) > 0$ for all $x \in \mathbb{R}$.*

*Indeed, suppose that the opposite statement holds:* $\mathrm{supp}\nu \subset \{0, 1, \ldots, K\}$

*for some finite $K \geqslant 1$. In this case, for $x > 0$,*

$$\overline{F}_{S_\nu}(x) = \sum_{n=1}^{K} \mathbb{P}(S_n > x)\mathbb{P}(\nu = n) = \sum_{n=1}^{K} \mathbb{P}(\xi_1 + \ldots + \xi_n > x)\mathbb{P}(\nu = n).$$

*For each $1 \leqslant n \leqslant K$,*

$$\mathbb{P}(\xi_1 + \ldots + \xi_n > x) \leqslant \mathbb{P}\bigg(\bigcup_{k=1}^{n}\{\xi_k > \frac{x}{n}\}\bigg) \leqslant n\mathbb{P}\bigg(\xi > \frac{x}{n}\bigg) \leqslant n\overline{F}_\xi\bigg(\frac{x}{K}\bigg).$$

*Hence, for all $x > 0$,*

$$\overline{F}_{S_\nu}(x) \leqslant \overline{F}_\xi\bigg(\frac{x}{K}\bigg)\sum_{n=1}^{K} n = \frac{K(K+1)}{2}\overline{F}_\xi\bigg(\frac{x}{K}\bigg),$$

*implying that*

$$\overline{F}_\xi(x) \geqslant \frac{2}{K(K+1)}\overline{F}_{S_\nu}(xK), \quad x > 0,$$

*and*

$$\begin{aligned}\liminf_{x\to\infty}\frac{\overline{F}_\xi(x)}{\overline{F}_{S_\nu}(x)} &\geqslant \frac{2}{K(K+1)}\liminf_{x\to\infty}\frac{\overline{F}_{S_\nu}(xK)}{\overline{F}_{S_\nu}(x)}\\ &= \frac{2}{K(K+1)}\Bigg(\limsup_{x\to\infty}\frac{\overline{F}_{S_\nu}(\frac{x}{K})}{\overline{F}_{S_\nu}(x)}\Bigg)^{-1} > 0\end{aligned}$$

*by the condition $F_{S_\nu} \in \mathscr{D}$. This resulting estimate contradicts the condition $\overline{F}_\xi(x) = o(\overline{F}_{S_\nu}(x))$, therefore the counting r.v. must have an infinite support.*

The following theorem generalizes Theorem 3.6.

**THEOREM 3.8.** *Suppose a sequence of r.v.s $\{\xi_1, \xi_2, \ldots\}$ and a counting r.v. $\nu$ satisfy the basic conditions of Theorem 3.7. If $F_{S_\nu} \in \mathscr{ERV}_{\{1,\beta\}}$ with $1 \leqslant \beta < \infty$, and $x\overline{F}_\xi(x) = o(\overline{F}_{S_\nu}(x))$, then $F_\nu \in \mathscr{D}$, and asymptotic inequalities* (3.2)-(3.4) *are satisfied with $\alpha = 1$.*

**REMARK 3.3.** *Conditions $F_{S_\nu} \in \mathscr{ERV}_{\{1,\beta\}}$ and $x\overline{F}_\xi(x) = o(\overline{F}_{S_\nu}(x))$ imply that $\mathbb{E}\xi$ is finite and, moreover, $\mathbb{E}(\xi)^r$ is also finite for some $r > 1$ (see Lemma 3.5 in [102] for details).*

Proofs of theorems provided in this section rely on three propositions formulated below. Proofs of these propositions follow a completely different approach compared to proofs of Theorems 3.5 and 3.6 given in [39]. Moreover, Propositions 2 and 3 present statements analogous to Theorems 3.5 and 3.6 but for the class $\mathscr{D}$ instead of $\mathscr{R}$. Detailed proofs of both theorems and these propositions are given in Section 3.3.

**PROPOSITION 3.1.** *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of i.i.d. copies of a nonnegative r.v. $\xi$ with positive mean $\mu = \mathbb{E}\xi$, and let $\nu$ be a counting r.v. with infinite support and independent of the sequence $\{\xi_1, \xi_2, \ldots\}$. Then*

$$\overline{F}_{S_\nu}\left((1-\varepsilon)\mu x\right) \gtrsim \overline{F}_\nu(x) \tag{3.5}$$

*for all $\varepsilon \in (0,1)$.*

**PROPOSITION 3.2.** *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of i.i.d. copies of a nonnegative r.v. $\xi$ with positive mean $\mu = \mathbb{E}\xi$ and d.f. $F_\xi$, and let $\nu$ be a counting r.v. with $\mathbb{E}\nu < \infty$ and independent of the sequence $\{\xi_1, \xi_2, \ldots\}$. If $F_{S_\nu} \in \mathscr{D}$ and $\overline{F}_\xi(x) = o(\overline{F}_{S_\nu}(x))$, then $F_\nu \in \mathscr{D}$, and*

$$\overline{F}_{S_\nu}\left((1+\varepsilon)\mu x\right) \lesssim \overline{F}_\nu(x) \tag{3.6}$$

*for all $\varepsilon > 0$.*

**PROPOSITION 3.3.** *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of i.i.d. copies of a nonnegative r.v. $\xi$ with positive mean $\mu = \mathbb{E}\xi$, finite moment $\mathbb{E}\xi^r$ of order $r > 1$, and d.f. $F_\xi$. Let $\nu$ be a counting r.v. independent of the sequence $\{\xi_1, \xi_2, \ldots\}$. If $F_{S_\nu} \in \mathscr{D}$ and $x\overline{F}_\xi(x) = o(\overline{F}_{S_\nu}(x))$, then $F_\nu \in \mathscr{D}$, and asymptotic relation (3.6) holds.*

## 3.3 Proofs of the Main Results

### 3.3.1 Auxiliary Lemmas

This subsection provides five auxiliary statements used in proofs of Propositions 3.2 and 3.3. The first three are required for the proof of Proposition 3.2, whereas the remaining two are used to prove Proposition 3.3. The subsection begins with a lemma that contains a construction of a special sequence of r.v.s with dominatedly varying distributions.

**LEMMA 3.1.** *Let $\{X_1, X_2, \ldots\}$ be a sequence of nonnegative i.i.d. r.v.s with common d.f. $F$ such that $\overline{F}(x) = o(\overline{H}(x))$ for some d.f. $H \in \mathscr{D}$. Then there exists a sequence of i.i.d. r.v.s $\{Z_1, Z_2, \ldots\}$ with common d.f. $G \in \mathscr{D}$ such that $\mathbb{P}(X_n \leqslant Z_n) = 1$ for $n \in \{1, 2, \ldots\}$ and $\overline{G}(x) = o(\overline{H}(x))$.*

*Proof.* The proof is based on the ideas of Lemma 4.4 from [39] and considerations in [133].

Let

$$
\begin{aligned}
&L(x) = 1 \quad \text{if} \quad x \in \left[0, x_0 = \max\left\{1, \sup\left\{z : \frac{\overline{F}(z)}{\overline{H}(z)} > 1\right\}\right\}\right],\\
&L(x) = 2 \quad \text{if} \quad x \in \left[x_0, x_1 = \max\left\{2x_0, \sup\left\{z : \frac{\overline{F}(z)}{\overline{H}(z)} > \frac{1}{2^2}\right\}\right\}\right],\\
&L(x) = 3 \quad \text{if} \quad x \in \left[x_1, x_2 = \max\left\{3x_1, \sup\left\{z : \frac{\overline{F}(z)}{\overline{H}(z)} > \frac{1}{3^2}\right\}\right\}\right],\\
&\qquad \ldots,\\
&L(x) = k \quad \text{if} \quad x \in \left[x_{k-2}, x_{k-1} = \max\left\{kx_{k-2}, \sup\left\{z : \frac{\overline{F}(z)}{\overline{H}(z)} > \frac{1}{k^2}\right\}\right\}\right],\\
&\qquad \ldots.
\end{aligned}
$$

According to the presented construction, when $x \to \infty$,

$$L(x) \to \infty \quad \text{and} \quad L(x)\frac{\overline{F}(x)}{\overline{H}(x)} \to 0,$$

because the sequence $\{x_k\}$ is unboundedly increasing and

$$L(x)\frac{\overline{F}(x)}{\overline{H}(x)} \leqslant \frac{k+1}{k^2}, \quad x \in (x_{k-1}, x_k],$$

for $k \in \{1, 2, \ldots\}$.

In addition, the function $L$ slowly varies because for each fixed $y > 1$ and large $x$, either both points $x$ and $yx$ belong to the same interval $(x_{k-1}, x_k]$, or $x \in (x_{k-1}, x_k]$ and $yx \in (x_k, x_{k+1}]$. In the first case,

$$\frac{L(yx)}{L(x)} = 1,$$

and, in the second case,

$$\frac{L(yx)}{L(x)} = \frac{k+2}{k+1}.$$

In both cases,

$$\lim_{x\to\infty} \frac{L(yx)}{L(x)} = 1,$$

implying that the function $L$ slowly varies by the classical definition (see Section 1 in [12]).

Let $\{W_1, W_2, \ldots\}$ be a sequence of i.i.d. positive r.v.s independent of the sequence $\{X_1, X_2, \ldots\}$ with tail function

$$\overline{D}(x) = \begin{cases} \frac{\overline{H}(x)}{L(x)} & \text{if} \quad x \geqslant 1, \\ 1 & \text{if} \quad x < 1. \end{cases}$$

By the construction of functions $D$ and $L$,

$$D \in \mathscr{D}, \quad \overline{D}(x) = o(\overline{H}(x)), \quad \overline{F}(x) = o(\overline{D}(x)). \tag{3.7}$$

Now suppose that

$$Z_n = X_n + W_n, \quad n \in \mathbb{N}.$$

This sequence of r.v.s has the following properties:

(i) $\mathbb{P}(X_n \leqslant Z_n) = 1$. This is obvious because $W_n \geqslant 0$ almost surely due to the construction of r.v.s $W_n$, $n \in \mathbb{N}$.

(ii) The sequence $\{Z_1, Z_2, \ldots\}$ consists of i.i.d. r.v.s because for any real numbers $\{x_1, x_2, \ldots, x_n\}$, $n \in \mathbb{N}$,

$$\begin{aligned} &\mathbb{P}(X_1 + W_1 \leqslant x_1, ..., X_n + W_n \leqslant x_n) \\ &= \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \mathbb{P}(X_1 + y_1 \leqslant x_1, ..., X_n + y_n \leqslant x_n) \mathrm{d}\mathbb{P}(W_1 \leqslant y_1, ..., W_n \leqslant y_n) \\ &= \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \prod_{j=1}^{n} \mathbb{P}(X_j + y_j \leqslant x_j) \mathrm{d}F_{W_1}(y_1) \ldots \mathrm{d}F_{W_n}(y_n) \\ &= \prod_{j=1}^{n} \mathbb{P}(X_j + W_j \leqslant x_j). \end{aligned}$$

(iii) The d.f. $G(x) = \mathbb{P}(Z_1 \leqslant x)$ satisfies the condition $\overline{G}(x) = o(\overline{H}(x))$ because

$$\overline{G}(x) = \mathbb{P}(X_1 + W_1 > x) = \int\limits_{[0,\frac{x}{2}]} \overline{F}(x-y)\mathrm{d}D(y) + \int\limits_{(\frac{x}{2},\infty)} \overline{F}(x-y)\mathrm{d}D(y)$$

$$\leqslant \overline{F}\Big(\frac{x}{2}\Big)\mathbb{P}\Big(0 \leqslant W_1 \leqslant \frac{x}{2}\Big) + \overline{D}\Big(\frac{x}{2}\Big) \leqslant \overline{F}\Big(\frac{x}{2}\Big) + \overline{D}\Big(\frac{x}{2}\Big), \tag{3.8}$$

and, therefore,

$$\limsup_{x\to\infty} \frac{\overline{G}(x)}{\overline{H}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}\big(\frac{x}{2}\big)}{\overline{H}\big(\frac{x}{2}\big)} \frac{\overline{H}\big(\frac{x}{2}\big)}{\overline{H}(x)} + \limsup_{x\to\infty} \frac{\overline{D}\big(\frac{x}{2}\big)}{\overline{D}(x)} \frac{\overline{D}(x)}{\overline{H}(x)} = 0$$

by (3.7) and the conditions of the lemma.

(iv) Finally, the d.f. $G$ belongs to the class $\mathscr{D}$ because

$$\limsup_{x\to\infty} \frac{\overline{G}\big(\frac{x}{2}\big)}{\overline{G}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}\big(\frac{x}{4}\big) + \overline{D}\big(\frac{x}{4}\big)}{\mathbb{P}(X_1 + W_1 > x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}\big(\frac{x}{4}\big) + \overline{D}\big(\frac{x}{4}\big)}{\overline{D}(x)}$$
$$\leqslant \limsup_{x\to\infty} \frac{\overline{F}\big(\frac{x}{4}\big)\overline{D}\big(\frac{x}{4}\big)}{\overline{D}\big(\frac{x}{4}\big)\overline{D}(x)} + \limsup_{x\to\infty} \frac{\overline{D}\big(\frac{x}{4}\big)}{\overline{D}(x)} < \infty$$

by relations (3.7) and (3.8). This finishes proof of the lemma. □

The following lemma presents a special property of i.i.d. r.v.s distributed according to the dominatedly varying distribution. This exceptional property plays an essential role in the proof of Proposition 3.2.

**LEMMA 3.2.** *Let $F \in \mathscr{D}$ be a d.f. on $\mathbb{R}$ with finite mean*

$$m = \int\limits_{-\infty}^{\infty} y\mathrm{d}F(y).$$

*Then for all $\gamma > \max\{0, m\}$, there exists a constant $D(\gamma) > 0$, independent of $x$ and $n$, such that*

$$\overline{F^{*n}}(x) \leqslant D(\gamma)n\overline{F}(x)$$

*for all $x \geqslant \gamma n$ and $n \in \mathbb{N}$.*

The proof of the lemma is presented in Theorem 1 of [104] and is generalized for a dependence structure in Corollary 3.1 of [101].

**REMARK 3.4.** *If a d.f. $F$ belongs to the class $\mathscr{D}$, then $F(x) < 1$ or, equivalently, $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$. This follows from the definition of class $\mathscr{D}$ because the condition*

$$\limsup_{x\to\infty} \frac{\overline{F}\left(\frac{x}{2}\right)}{\overline{F}(x)} < \infty$$

*cannot be satisfied in the case $\overline{F}(x_0) = 0$ for some $x_0 \in \mathbb{R}$.*

The following lemma is another version of Lemma 3.2.

**LEMMA 3.3.** *Let $F$ be a d.f. on $\mathbb{R}$ with finite mean $m$. Then for all $\gamma > \max\{0, m\}$, there exists a positive constant $D(\gamma)$, independent of $x$ and $n$, such that*

$$\max_{1\leqslant n\leqslant \frac{x}{\gamma}} \frac{\overline{F^{*n}}(x)}{n\overline{F}(x)} \leqslant D(\gamma)$$

*for $x \geqslant \gamma$.*

*Proof.* By Lemma 3.2,

$$\begin{aligned}
x \geqslant \gamma \quad &\Longrightarrow \quad \frac{\overline{F^{*1}}(x)}{\overline{F}(x)} \leqslant D(\gamma),\\
x \geqslant 2\gamma \quad &\Longrightarrow \quad \frac{\overline{F^{*2}}(x)}{2\overline{F}(x)} \leqslant D(\gamma),\\
x \geqslant 3\gamma \quad &\Longrightarrow \quad \frac{\overline{F^{*3}}(x)}{3\overline{F}(x)} \leqslant D(\gamma),\\
&\dots\\
x \geqslant N\gamma \quad &\Longrightarrow \quad \frac{\overline{F^{*N}}(x)}{N\overline{F}(x)} \leqslant D(\gamma).
\end{aligned}$$

For $x \geqslant \gamma$, it holds that $x \geqslant \left\lfloor \frac{x}{\gamma} \right\rfloor \gamma$, hence, the above estimates imply that

$$\max_{1\leqslant n\leqslant \lfloor\frac{x}{\gamma}\rfloor} \frac{\overline{F^{*n}}(x)}{n\overline{F}(x)} \leqslant D(\gamma).$$

The lemma is proved. □

The following lemma gives an important inequality used in the proof of Proposition 3.3.

**LEMMA 3.4.** *Let $\{X_1, X_2, \ldots\}$ be independent copies of a r.v. $X$ with d.f. $F$, mean $0$, and $\mathbb{E}(X^+)^r < \infty$ for some $r > 1$. Then for all $\gamma > 0$ and $p > 0$, there exist positive numbers $a$ and $C = C(\gamma, p)$, independent of $x$ and $n$, such that*

$$\mathbb{P}\biggl(\sum_{k=1}^{n} X_k > x\biggr) \leqslant n\overline{F}(ax) + \frac{C}{x^p}$$

*for all $x \geqslant \gamma n$ and $n \in \mathbb{N}$.*

The above lemma is proved in [101] (see Lemma 2.3) for possibly dependent identically distributed r.v.s. Discussions on similar inequalities, which can also be used in the proof of Proposition 3.3, can be found in [44, 80].

The last lemma is related to the tail property of distributions from the class $\mathscr{D}$ which is used in the proof of Proposition 3.3.

**LEMMA 3.5.** *Let $F \in \mathscr{D}$ be a distribution with upper Matuszewska index $J_F^+$. Then for all $p > J_F^+$, it holds that $x^{-p} = o(\overline{F}(x))$.*

It follows from Proposition 2.2.1 in [12]. Moreover, some details of the proof can be found in Lemma 3.5 of [102].

### 3.3.2 Proof of Proposition 3.1

Since the generating r.v. $\xi$ is nonnegative, for all $x > 0$ and $\varepsilon \in (0, 1)$,

$$\begin{aligned}
\overline{F}_{S_\nu}((1-\varepsilon)\mu x) &= \sum_{n=1}^{\infty} \mathbb{P}(S_n > (1-\varepsilon)\mu x)\mathbb{P}(\nu = n) \\
&\geqslant \sum_{n>x} \mathbb{P}\bigl(S_n > (1-\varepsilon)\mu x\bigr)\mathbb{P}(\nu = n) \\
&\geqslant \sum_{n>x} \mathbb{P}\bigl(S_{\lfloor x\rfloor} > (1-\varepsilon)\mu(\lfloor x\rfloor + \{x\})\bigr)\mathbb{P}(\nu = n) \\
&= \mathbb{P}\biggl(\frac{S_{\lfloor x\rfloor} - \mu\lfloor x\rfloor}{\lfloor x\rfloor} > -\varepsilon\mu + (1-\varepsilon)\mu\frac{\{x\}}{\lfloor x\rfloor}\biggr)\overline{F}_\nu(x).
\end{aligned}$$

According to conditions of the proposition, $\overline{F}_\nu(x) > 0$ for all $x \in \mathbb{R}$, hence,

$$\liminf_{x\to\infty} \frac{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)}{\overline{F}_\nu(x)}$$
$$\geqslant \liminf_{x\to\infty} \mathbb{P}\left(\frac{S_{\lfloor x\rfloor} - \mu\lfloor x\rfloor}{\lfloor x\rfloor} > -\varepsilon\mu + (1-\varepsilon)\mu\frac{\{x\}}{\lfloor x\rfloor}\right) = 1$$

by the law of large numbers. Proposition 3.1 is proved. □

### 3.3.3 Proof of Proposition 3.2

For all $x > 1$ and $\varepsilon > 0$,

$$\overline{F}_{S_\nu}((1+\varepsilon)\mu x) = \left(\sum_{1\leqslant n\leqslant x} + \sum_{n>x}\right)\mathbb{P}(S_n > (1+\varepsilon)\mu x)\mathbb{P}(\nu = n)$$
$$=: \mathcal{J}_1(x,\varepsilon) + \mathcal{J}_2(x,\varepsilon) \tag{3.9}$$

For the term $\mathcal{J}_2(x,\varepsilon)$,

$$\frac{\mathcal{J}_2(x,\varepsilon)}{\overline{F}_\nu(x)} \leqslant \sup_{n>x} \mathbb{P}(S_n > (1+\varepsilon)\mu x) \leqslant 1 \tag{3.10}$$

because $\overline{F}_\nu(x) > 0$ for all $x \in \mathbb{R}$ due to Remark 3.2.

Now let $\{\hat{\xi}_1, \hat{\xi}_2, \ldots\}$ be a sequence of nonnegative i.i.d. r.v.s with properties $\mathbb{P}(\xi_k \leqslant \hat{\xi}_k) = 1$ for $k \in \{1, 2, \ldots\}$, $F_{\hat{\xi}_1} \in \mathscr{D}$, and $\overline{F}_{\hat{\xi}_1}(x) = o(\overline{F}_{S_\nu}(x))$. Such construction is possible due to Lemma 3.1.

For $\hat{S}_n = \sum\limits_{k=1}^{n} \hat{\xi}_k$ and $x > \frac{\hat{\mu}}{\mu}$,

$$\mathcal{J}_1(x,\varepsilon) \leqslant \sum_{1\leqslant n\leqslant \frac{\mu x}{\hat{\mu}}} \mathbb{P}(\hat{S}_n > (1+\varepsilon)\mu x)\mathbb{P}(\nu = n)$$
$$+ \sum_{\frac{\mu x}{\hat{\mu}} < n \leqslant x} \mathbb{P}(S_n > (1+\varepsilon)\mu x)\mathbb{P}(\nu = n)$$
$$=: \mathcal{J}_{11}(x,\varepsilon) + \mathcal{J}_{12}(x,\varepsilon) \tag{3.11}$$

where $\mu \leqslant \hat{\mu} = \mathbb{E}\hat{\xi}_1 < \infty$ because $\mathbb{E}S_\nu = \mu\nu < \infty$ by the conditions of the proposition, and $\overline{F}_{\hat{\xi}_1}(x) = o(\overline{F}_{S_\nu}(x))$ by the construction of r.v.s

$\{\hat{\xi}_1, \hat{\xi}_2, \ldots\}$. It is clear that

$$\begin{aligned}
\mathcal{J}_{11}(x,\varepsilon) &= \sum_{1\leqslant n\leqslant \frac{\mu x}{\hat{\mu}}} \mathbb{P}(\hat{S}_n - n\hat{\mu} > -n\hat{\mu} + \mu x + \varepsilon\mu x)\mathbb{P}(\nu = n) \\
&\leqslant \sum_{1\leqslant n\leqslant \frac{\mu x}{\hat{\mu}}} \mathbb{P}(\hat{S}_n - n\hat{\mu} > \varepsilon\mu x)\mathbb{P}(\nu = n).
\end{aligned}$$

By using Lemma 3.3 for r.v.s $\{\hat{\xi}_1 - \hat{\mu}, \hat{\xi}_2 - \hat{\mu}, \ldots\}$, one can obtain that

$$\begin{aligned}
\mathcal{J}_{11}(x,\varepsilon) \leqslant& \sum_{1\leqslant n\leqslant \frac{\varepsilon\mu x}{\varepsilon\hat{\mu}}} \mathbb{P}(\hat{S}_n - n\hat{\mu} > \varepsilon\mu x)\mathbb{P}(\nu = n) \\
\leqslant& C_1\overline{F}_{\hat{\xi}_1-\hat{\mu}}(\varepsilon\mu x) \sum_{1\leqslant n\leqslant \frac{\mu x}{\hat{\mu}}} n\mathbb{P}(\nu = n) \leqslant C_1\mathbb{E}\nu\overline{F}_{\hat{\xi}_1}(\varepsilon\mu x)
\end{aligned}$$

for $x \geqslant \varepsilon\hat{\mu}$ and some positive constant $C_1 = C_1(\varepsilon, \hat{\mu})$. The last estimate implies that

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\mathcal{J}_{11}(x,\varepsilon)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} &\leqslant C_1\mathbb{E}\nu \limsup_{x\to\infty} \frac{\overline{F}_{\hat{\xi}_1}(\varepsilon\mu x)}{\overline{F}_{\hat{\xi}_1}(x)} \\
&\times \limsup_{x\to\infty} \frac{\overline{F}_{\hat{\xi}_1}(x)}{\overline{F}_{S_\nu}(x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} = 0 \qquad (3.12)
\end{aligned}$$

because

$$\limsup_{x\to\infty} \frac{\overline{F}_{\hat{\xi}_1}(\varepsilon\mu x)}{\overline{F}_{\hat{\xi}_1}(x)} < \infty$$

by the condition $F_{\hat{\xi}_1} \in \mathscr{D}$,

$$\limsup_{x\to\infty} \frac{\overline{F}_{\hat{\xi}_1}(x)}{\overline{F}_{S_\nu}(x)} = 0$$

by the construction of r.v.s $\{\hat{\xi}_1, \hat{\xi}_2, \ldots\}$, and

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} < \infty$$

by the condition $F_{S_\nu} \in \mathscr{D}$.

Further, consider the term $\mathcal{J}_{12}(x,\varepsilon)$. It is evident that

$$\mathcal{J}_{12}(x,\varepsilon) = \sum_{\frac{\mu x}{\hat{\mu}} < n \leqslant x} \mathbb{P}(S_n > (1+\varepsilon)\mu x)\mathbb{P}(\nu = n)$$

$$= \sum_{\frac{\mu x}{\hat{\mu}} < n \leqslant x} \mathbb{P}(S_n - n\mu > -n\mu + \mu x + \varepsilon\mu x)\mathbb{P}(\nu = n)$$

$$\leqslant \sum_{\frac{\mu x}{\hat{\mu}} < n \leqslant x} \mathbb{P}(S_n - n\mu > \varepsilon\mu x)\mathbb{P}(\nu = n)$$

$$\leqslant \sup_{n > \frac{\mu x}{\hat{\mu}}} \mathbb{P}\left(\frac{S_n}{n} - \mu > \varepsilon\mu\right)\overline{F}_\nu\left(\frac{\mu x}{\hat{\mu}}\right). \tag{3.13}$$

Now let $\delta \in (0,1)$. The law of large numbers, decompositions (3.9) and (3.11), and estimates (3.10), (3.12), and (3.13) imply that

$$\overline{F}_{S_\nu}((1+\varepsilon)\mu x) \leqslant \delta\overline{F}_{S_\nu}((1+\varepsilon)\mu x) + \delta\overline{F}_\nu\left(\frac{\mu x}{\hat{\mu}}\right) + \overline{F}_\nu(x) \tag{3.14}$$

for sufficiently large $x$, say $x \geqslant x_1(\varepsilon,\delta)$. In the particular case $\varepsilon = \delta = \frac{1}{2}$, for $x \geqslant \hat{x}_1 = x_1(\frac{1}{2},\frac{1}{2})$, one can get

$$\overline{F}_{S_\nu}\left(\frac{3}{2}\mu x\right) \leqslant 3\overline{F}_\nu\left(\frac{\mu x}{\hat{\mu}}\right).$$

Meanwhile, from the estimate of Proposition 3.1,

$$\overline{F}_{S_\nu}\left(\frac{\mu x}{2}\right) \geqslant \frac{1}{2}\overline{F}_\nu(x)$$

for sufficiently large $x$, say $x \geqslant x_2$. The last two inequalities imply $F_\nu \in \mathscr{D}$ because

$$\frac{\overline{F}_\nu\left(\frac{x}{2}\right)}{\overline{F}_\nu(x)} \leqslant \frac{6\overline{F}_{S_\nu}\left(\frac{\mu x}{4}\right)}{\overline{F}_{S_\nu}\left(\frac{3\hat{\mu}x}{2}\right)}$$

for $x \geqslant \max\{\frac{\mu\hat{x}_1}{\hat{\mu}}, 2x_2\}$ and $F_{S_\nu} \in \mathscr{D}$ by the conditions of the proposition. Now from estimate (3.14) it follows that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \leqslant \frac{1}{1-\delta}\left(\delta \limsup_{x\to\infty} \frac{\overline{F}_\nu(\frac{\mu x}{\hat{\mu}})}{\overline{F}_\nu(x)} + 1\right)$$

for all $\varepsilon > 0$ and $\delta \in (0,1)$. The last estimate finishes the proof of

Proposition 3.2 because $F_\nu \in \mathscr{D}$ and $\delta$ can be chosen arbitrarily small.

$\square$

### 3.3.4 Proof of Proposition 3.3

According to decomposition (3.9), for all $x > 1$ and $\varepsilon > 0$,

$$\overline{F}_{S_\nu}((1+\varepsilon)\mu x) \leqslant \mathcal{J}_1(x) + \overline{F}_\nu(x) \tag{3.15}$$

with

$$\mathcal{J}_1(x) = \sum_{1\leqslant n\leqslant x} \mathbb{P}(S_n > (1+\varepsilon)\mu x)\mathbb{P}(\nu = n) \leqslant \mathbb{P}(S_{\lfloor x\rfloor} > (1+\varepsilon)\mu x)$$

$$\leqslant \mathbb{P}\Big(\bigcup_{n\leqslant\lfloor x\rfloor}\{\xi_n > \varepsilon x\}\Big) + \mathbb{P}\Big(\bigcap_{n\leqslant\lfloor x\rfloor}\{\xi_n \leqslant \varepsilon x\}, S_{\lfloor x\rfloor} > (1+\varepsilon)\mu x\Big)$$

$$=: \mathcal{K}_1(x) + \mathcal{K}_2(x). \tag{3.16}$$

Since $F_{S_\nu} \in \mathscr{D}$, the conditions of Proposition 3.3 imply that

$$\limsup_{x\to\infty} \frac{\mathcal{K}_1(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} \leqslant \limsup_{x\to\infty} \frac{x\overline{F}_\xi(\varepsilon x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}$$

$$\frac{1}{\varepsilon}\limsup_{x\to\infty} \frac{\varepsilon x\overline{F}_\xi(\varepsilon x)}{\overline{F}_{S_\nu}(\varepsilon x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(\varepsilon x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} = 0. \tag{3.17}$$

For the second term in (3.16),

$$\mathcal{K}_2(x) = \mathbb{P}\Big(\bigcap_{n\leqslant\lfloor x\rfloor}\{\xi_n \leqslant \varepsilon x\}, \sum_{n=1}^{\lfloor x\rfloor}(\xi_n\mathbb{1}_{\{\xi_n\leqslant\varepsilon x\}} + \xi_n\mathbb{1}_{\{\xi_n>\varepsilon x\}}) > (1+\varepsilon)\mu x\Big)$$

$$= \mathbb{P}\Big(\bigcap_{n\leqslant\lfloor x\rfloor}\{\xi_n \leqslant \varepsilon x\}, \sum_{n=1}^{\lfloor x\rfloor}\xi_n\mathbb{1}_{\{\xi_n\leqslant\varepsilon x\}} > (1+\varepsilon)\mu x\Big)$$

$$\leqslant \mathbb{P}\Big(\sum_{n=1}^{\lfloor x\rfloor}\xi_n\mathbb{1}_{\{\xi_n\leqslant\varepsilon x\}} > (1+\varepsilon)\mu x\Big)$$

$$= \mathbb{P}\Big(\sum_{n=1}^{\lfloor x\rfloor}(\xi_n\mathbb{1}_{\{\xi_n\leqslant\varepsilon x\}} - \mathbb{E}\xi_n\mathbb{1}_{\{\xi_n\leqslant\varepsilon x\}}) > (1+\varepsilon)\mu x - \sum_{n=1}^{\lfloor x\rfloor}\mathbb{E}\xi_n\mathbb{1}_{\{\xi_n\leqslant\varepsilon x\}}\Big)$$

$$\leqslant \mathbb{P}\bigg(\sum_{n=1}^{\lfloor x\rfloor}(\xi_n \mathbb{1}_{\{\xi_n\leqslant \varepsilon x\}} - \mathbb{E}\xi_n \mathbb{1}_{\{\xi_n\leqslant \varepsilon x\}}) > \varepsilon\mu x\bigg)$$

because, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\lfloor x\rfloor}\mathbb{E}\xi_n \mathbb{1}_{\{\xi_n\leqslant \varepsilon x\}} \leqslant \mu x.$$

R.v.s $Y_n := \xi_n \mathbb{1}_{\{\xi_n\leqslant \varepsilon x\}} - \mathbb{E}\xi_n \mathbb{1}_{\{\xi_n\leqslant \varepsilon x\}}$, $n \in \mathbb{N}$, satisfy conditions of Lemma 3.4, that is, the sequence $\{Y_1, Y_2, \dots\}$ consists of i.i.d. r.v.s, $\mathbb{E}Y_1 = 0$, and

$$\mathbb{E}(Y_1^+)^r = \mathbb{E}((\xi\mathbb{1}_{\{\xi\leqslant\varepsilon x\}} - \mathbb{E}\xi\mathbb{1}_{\{\xi\leqslant\varepsilon x\}})^+)^r \leqslant \mathbb{E}((\xi\mathbb{1}_{\{\xi\leqslant\varepsilon x\}})^+)^r \leqslant \mathbb{E}\xi^r < \infty$$

for $r > 1$ by conditions of Proposition 3.3. By Lemma 3.4, with $\gamma = \frac{\varepsilon\mu}{2}$ and $p > J^+_{F_{S_\nu}}$, it holds that

$$\mathcal{K}_2(x) \leqslant \mathbb{P}\bigg(\sum_{n=1}^{\lfloor x\rfloor} Y_n > \varepsilon\mu x\bigg) \leqslant \lfloor x\rfloor \overline{F}_{Y_1}(a\varepsilon\mu x) + \frac{C_2}{(\varepsilon\mu)^p x^p} \tag{3.18}$$

with constants $a$ and $C_2$ depending on $\mathbb{E}(Y_1^+)^r$, $\varepsilon$, and $p$, but independent of $x$. By the definition of r.v. $Y_1$, for $x > 0$,

$$\begin{aligned}
\overline{F}_{Y_1}(a\varepsilon\mu x) &= \mathbb{P}(Y_1 > a\varepsilon\mu x) = \mathbb{P}(\xi\mathbb{1}_{\{\xi\leqslant\varepsilon x\}} > a\varepsilon\mu x + \mathbb{E}\xi\mathbb{1}_{\{\xi\leqslant\varepsilon x\}}) \\
&\leqslant \mathbb{P}(\xi\mathbb{1}_{\{\xi\leqslant\varepsilon x\}} > a\varepsilon\mu x, \xi \leqslant \varepsilon x) + \mathbb{P}(\xi\mathbb{1}_{\{\xi\leqslant\varepsilon x\}} > a\varepsilon\mu x, \xi > \varepsilon x) \\
&\leqslant \mathbb{P}(\xi > a\varepsilon\mu x) = \overline{F}_\xi(a\varepsilon\mu x).
\end{aligned}$$

Therefore, by estimate (3.18), conditions $x\overline{F}_\xi(x) = o(\overline{F}_{S_\nu}(x))$ and $F_{S_\nu} \in \mathscr{D}$, and Lemma 3.5, one can derive

$$\limsup_{x\to\infty}\frac{\mathcal{K}_2(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}$$

$$\leqslant \frac{1}{a\varepsilon\mu}\limsup_{x\to\infty}\frac{a\varepsilon\mu x\overline{F}_\xi(a\varepsilon\mu x)}{\overline{F}_{S_\nu}(a\varepsilon\mu x)}\limsup_{x\to\infty}\frac{\overline{F}_{S_\nu}(a\varepsilon\mu x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}$$

$$+\frac{C_2}{(\varepsilon\mu)^p}\limsup_{x\to\infty}\frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}\limsup_{x\to\infty}\frac{1}{x^p\overline{F}_{S_\nu}(x)} = 0.$$

Now let $\delta \in (0, 1)$. Decompositions (3.15) and (3.16) and relations (3.17) and (3.18) imply that

$$\overline{F}_{S_\nu}((1+\varepsilon)\mu x) \leqslant \delta \overline{F}_{S_\nu}((1+\varepsilon)\mu x) + \overline{F}_\nu(x) \tag{3.19}$$

for sufficiently large $x$, say $x \geqslant x_3(\varepsilon, \delta)$. In this case $\varepsilon = \delta = \frac{1}{2}$, one can get

$$\frac{1}{2}\overline{F}_{S_\nu}\left(\frac{3}{2}\mu x\right) \leqslant \overline{F}_\nu(x)$$

for $x \geqslant x_3(\frac{1}{2}, \frac{1}{2})$. Meanwhile, Proposition 3.1 implies

$$\overline{F}_{S_\nu}\left(\frac{1}{2}\mu x\right) \geqslant \frac{1}{2}\overline{F}_\nu(x)$$

for large $x$, say $x \geqslant x_4$. The last two estimates show that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \leqslant \frac{1}{1-\delta}$$

for all $\varepsilon > 0$ and $\delta \in (0, 1)$. Hence,

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \leqslant 1$$

due to the arbitrary choice of parameter $\delta \in (0, 1)$. This finishes the proof of the proposition. □

### 3.3.5 Proof of Theorem 3.7

Let $F_{S_\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$, and suppose that $\mu = \mathbb{E}\xi > 1$. In such case, $(1-\varepsilon)\mu > 1$ if $\varepsilon \in (0, 1)$ is sufficiently small. Since $\mathscr{ERV} \subset \mathscr{D}$, all conditions of Propositions 3.1 and 3.2 are satisfied. By Proposition 3.1,

$$\limsup_{x\to\infty} \frac{\overline{F}_\nu(x)}{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)} \leqslant 1,$$

and the condition $F_{S_\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$ implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)}{\overline{F}_{S_\nu}(x)} \leqslant ((1-\varepsilon)\mu)^{-\alpha}$$

for $\varepsilon \in (0, 1 - \frac{1}{\mu})$. Therefore, for such $\varepsilon$,

$$\limsup_{x\to\infty} \frac{\mu^\alpha \overline{F}_\nu(x)}{\overline{F}_{S_\nu}(x)} \leqslant \mu^\alpha \limsup_{x\to\infty} \frac{\overline{F}_\nu(x)}{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)}{\overline{F}_{S_\nu}(x)}$$

$$\leqslant (1-\varepsilon)^{-\alpha},$$

implying that

$$\mu^\alpha \overline{F}_\nu(x) \lesssim \overline{F}_{S_\nu}(x). \tag{3.20}$$

Due to Proposition 3.2,

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \leqslant 1$$

for $\varepsilon > 0$, and the condition $F_{S_\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$ implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} = \left( \liminf_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_{S_\nu}(x)} \right)^{-1} \leqslant ((1+\varepsilon)\mu)^\beta.$$

Therefore

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\mu^\beta \overline{F}_\nu(x)} \leqslant \frac{1}{\mu^\beta} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)}$$

$$\leqslant (1+\varepsilon)^\beta,$$

implying that

$$\overline{F}_{S_\nu}(x) \lesssim \mu^\beta \overline{F}_\nu(x) \tag{3.21}$$

Relations (3.20) and (3.21) imply the first asymptotic inequality (3.2) of Theorem 3.7.

Now let again $F_{S_\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$, but let $\mu = \mathbb{E}\xi < 1$. In such case, $\frac{1}{(1+\varepsilon)\mu} > 1$ if $\varepsilon \in (0,1)$ is sufficiently small. Since $\mathscr{ERV} \subset \mathscr{D}$, all conditions of Propositions 3.1 and 3.2 are satisfied. By Proposition 3.2,

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \leqslant 1,$$

and the condition $F_{S_\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$ implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} = \limsup_{z\to\infty} \frac{\overline{F}_{S_\nu}(\frac{z}{(1+\varepsilon)\mu})}{\overline{F}_{S_\nu}(z)} \leqslant ((1+\varepsilon)\mu)^\alpha$$

if $\varepsilon \in (0, \dfrac{1}{\mu} - 1)$. Consequently,

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\mu^\alpha \overline{F}_\nu(x)} \leqslant \frac{1}{\mu^\alpha} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}$$
$$\leqslant (1+\varepsilon)^\alpha,$$

and

$$\overline{F}_{S_\nu}(x) \lesssim \mu^\alpha \overline{F}_\nu(x) \tag{3.22}$$

due to the arbitrariness of $\varepsilon \in (0, \dfrac{1}{\mu} - 1)$.

If $\mu < 1$, then $\frac{1}{(1+\varepsilon)\mu} > 1$ for all $\varepsilon \in (0,1)$. Hence the condition $F_{S_\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$ implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)}{\overline{F}_{S_\nu}(x)} = \limsup_{z\to\infty} \frac{\overline{F}_{S_\nu}(z)}{\overline{F}_{S_\nu}(\frac{z}{(1-\varepsilon)\mu})}$$
$$= \left( \liminf_{z\to\infty} \frac{\overline{F}_{S_\nu}(\frac{z}{(1-\varepsilon)\mu})}{\overline{F}_{S_\nu}(z)} \right)^{-1} \leqslant ((1-\varepsilon)\mu)^{-\beta}.$$

By Proposition 3.1,

$$\limsup_{x\to\infty} \frac{\mu^\beta \overline{F}_\nu(x)}{\overline{F}_{S_\nu}(x)} \leqslant \mu^\beta \limsup_{x\to\infty} \frac{\overline{F}_\nu(x)}{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)}{\overline{F}_{S_\nu}(x)}$$
$$\leqslant (1-\varepsilon)^{-\beta},$$

which implies that

$$\mu^\beta \overline{F}_\nu(x) \lesssim \overline{F}_{S_\nu}(x). \tag{3.23}$$

Asymptotic inequalities (3.22) and (3.23) imply the second relation (3.3) of the theorem.

Consider the last case $\mu = \mathbb{E}\xi = 1$. Since $F_{S_\nu} \in \mathscr{ERV} \subset \mathscr{D}$ and $\mu = 1$, Propositions 3.1 and 3.2 imply that

$$\overline{F}_{S_\nu}((1+\varepsilon)x) \lesssim \overline{F}_\nu(x) \lesssim \overline{F}_{S_\nu}((1-\varepsilon)x)$$

for all $\varepsilon \in (0,1)$. By the definition of the class $\mathscr{ERV}_{\{\alpha,\beta\}}$,

$$\overline{F}_{S_\nu}((1+\varepsilon)x) \gtrsim (1+\varepsilon)^{-\beta} \overline{F}_{S_\nu}(x)$$

and

$$\overline{F}_{S_\nu}((1-\varepsilon)x) \lesssim (1-\varepsilon)^{-\beta}\overline{F}_{S_\nu}(x).$$

Consequently, for all $\varepsilon \in (0,1)$,

$$(1+\varepsilon)^{-\beta}\overline{F}_{S_\nu}(x) \lesssim \overline{F}_\nu(x) \lesssim (1-\varepsilon)^{-\beta}\overline{F}_{S_\nu}(x),$$

which implies relation (3.4). Theorem 3.7 is proved. $\square$

### 3.3.6 Proof of Theorem 3.8

The proof of Theorem 3.8 is completely analogous to that of Theorem 3.7. The only modification is the use of Proposition 3.3 instead of Proposition 3.2 since the conditions of Theorem 3.8 are consistent with those of Proposition 3.3. $\square$

## 3.4 Example

This section presents one illustrative example for the asymptotic relations stated in Theorems 3.6 and 3.8.

Consider the following model. The sequence of r.v.s $\{\xi_1, \xi_2, \ldots\}$ consists of independent copies of a r.v. $\xi$ distributed according to the exponential law, that is,

$$\overline{F}_\xi(x) = \mathbb{P}(\xi > x) = e^{-x}, \quad x \geqslant 0.$$

The counting r.v. $\nu$ independent of the sequence $\{\xi_1, \xi_2, \ldots\}$ is distributed according to the zeta law, that is,

$$\mathbb{P}(\nu = n) = \frac{1}{\zeta(2)}\frac{1}{n^2}, \quad n \in \{1, 2, \ldots\}, \quad \text{where } \zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

In this case, $\mathbb{E}\xi = 1$, $\mathbb{E}\xi^r < \infty$ for all $r > 1$, $\mathbb{E}\nu = \infty$, and

$$\limsup_{x\to\infty}\frac{x\overline{F}_\xi(x)}{\overline{F}_{S_\nu}(x)} = \limsup_{x\to\infty}\frac{xe^{-x}}{\frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{1}{n^2}\overline{F_\xi^{*n}}(x)}$$

$$\leqslant \limsup_{x\to\infty}\frac{9\zeta(2)xe^{-x}}{(1+x+\frac{x^2}{2!})e^{-x}} = 0.$$

In addition, for $x > 0$,

$$\overline{F}_{S_\nu}(x) = \frac{1}{\zeta(2)} \sum_{n=1}^{\infty} \frac{1}{n^2} \overline{F_\xi^{*n}}(x) = \frac{1}{\zeta(2)} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}$$

$$= \frac{e^{-x}}{\zeta(2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=k+1}^{\infty} \frac{1}{n^2}.$$

Since

$$\frac{1}{k+1} \leqslant \sum_{n=k+1}^{\infty} \frac{1}{n^2} \leqslant \frac{1}{k}$$

for all $k \geqslant 1$,

$$\overline{F}_{S_\nu}(x) = \frac{e^{-x}}{\zeta(2)} \left( \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{k=1}^{\infty} \frac{x^k}{k!} \sum_{n=k+1}^{\infty} \frac{1}{n^2} \right) \geqslant \frac{e^{-x}}{\zeta(2)} \left( 1 + \sum_{k=1}^{\infty} \frac{x^k}{(k+1)!} \right)$$

$$= \frac{e^{-x}}{\zeta(2)} \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} = \frac{e^{-x}}{\zeta(2)} \frac{e^x - 1}{x} \gtrsim \frac{1}{\zeta(2)x}$$

and

$$\overline{F}_{S_\nu}(x) \leqslant e^{-x} + \frac{e^{-x}}{\zeta(2)} \sum_{k=1}^{\infty} \frac{x^k}{k!k} = e^{-x} + \frac{e^{-x}}{\zeta(2)} \left( \sum_{k=1}^{\infty} \frac{x^k}{(k+1)!} + \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)!} \right)$$

$$\leqslant e^{-x} + \frac{e^{-x}}{\zeta(2)} \left( \frac{e^x - 1}{x} + \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)!} \right).$$

The last estimate implies that

$$\overline{F}_{S_\nu}(x) \lesssim \frac{1}{\zeta(2)x}$$

because

$$\frac{e^{-x} \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)!}}{\frac{1}{x}} = xe^{-x} \left( \sum_{k=1}^{K} \frac{x^k}{k(k+1)!} + \sum_{k=K+1}^{\infty} \frac{x^k}{k(k+1)!} \right)$$

$$\leqslant x^{K+1} e^{-x} \sum_{k=1}^{K} \frac{1}{k(k+1)!} + xe^{-x} \frac{e^x - 1}{x} \frac{1}{K}$$

for all $x > 0$ and an arbitrary $K \geqslant 2$.

It can be noticed from the obtained relations that

$$\overline{F}_{S_\nu}(x) \sim \frac{1}{\zeta(2)x} \sim \mathbb{E}\xi\overline{F}_\nu(x),$$

which is consistent with Theorems 3.6 and 3.8. The results of simulated relations are presented in Figure 3.1.

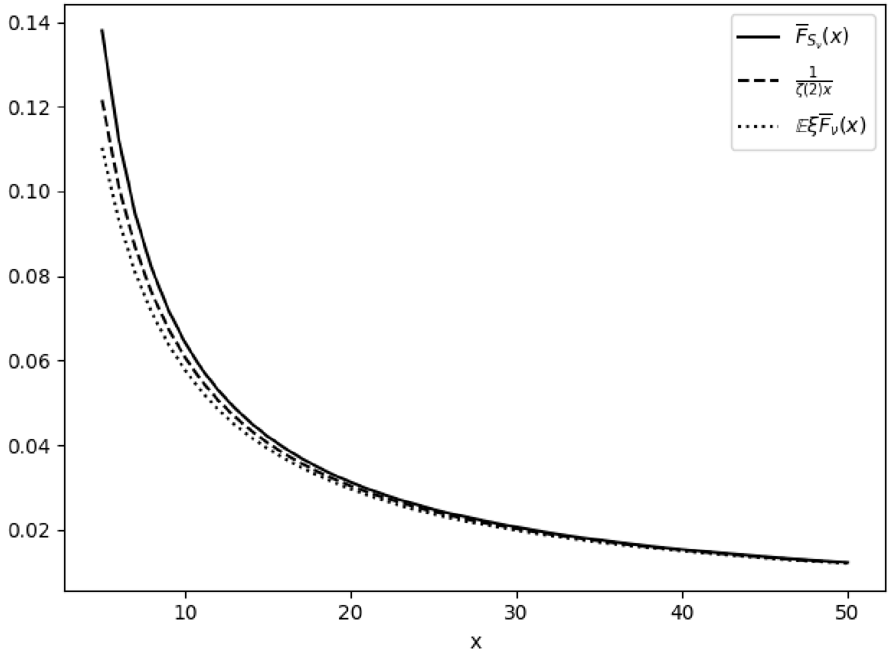


**FIGURE 3.1.** Simulated asymptotic relations

# 4 ASYMPTOTIC FORMULAS FOR THE HAEZENDONCK-GOOVAERTS RISK MEASURE

As in Chapter 2, this chapter considers sums with a fixed number of summands, namely $S_n^{\xi}$ and $S_n^{\theta\xi}$. Assuming that r.v.s $\xi_1, \ldots, \xi_n$ represent individual random losses, the analysis presented here is devoted to the assessment of risk associated with these aggregate loss models.

Problems of this type arise frequently in insurance and finance, particularly in the context of capital allocation, risk premium estimation, solvency assessment, and stress testing. Accurate evaluation of the risk inherent in aggregated losses is therefore of both theoretical and practical importance.

More precisely, this chapter focuses on the Haezendonck-Goovaerts (HG) risk measure (see Definition 1.19), which is particularly useful in actuarial science and financial risk management due to its ability to capture heavy-tailed and extreme losses. The HG risk measure is based on Orlicz space, allowing for flexible modelling of risk aversion and tail behaviour. This flexibility, and in particular the freedom to choose the associated Young function, provides greater control over the influence of extreme losses on the resulting risk assessment. At the same time, despite its solid theoretical foundations, the practical implementation of the HG risk measure may be challenging. Motivated by this, the aim of this chapter is to derive asymptotic estimations of the HG risk measure for sums $S_n^{\xi}$ and $S_n^{\theta\xi}$ as the confidence level tends to 1.

The rest of this chapter is structured as follows. Section 4.1 provides the necessary background and reviews known results. Section 4.2 formulates the main results of this chapter, followed by their proofs in Section 4.3, while Section 4.4 deals with the analysis of the particular example illustrating the accuracy of the derived asymptotic formula.

## 4.1 Background

The section begins by introducing an additional concept characterising the monotonicity of a function used in the main results of this chapter. This concept applies to a broad class of functions (see, e.g., [14, 17]). However, attention is restricted to the relevant subclass of d.f.s considered in [12, 64, 71] among others.

**DEFINITION 4.1.** *A distribution function $F$ supported on $\mathbb{R}$ is said to have a positively decreasing tail, denoted by $F \in \mathcal{PD}$, if for any fixed $y \in (0,1)$*

$$\liminf_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} > 1,$$

*or equivalently*

$$\limsup_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} < 1$$

*for any $y > 1$.*

It can be observed that

$$\mathscr{R}_+ := \bigcup_{\alpha>0} \mathscr{R}_\alpha \subset \mathcal{PD}; \quad \mathscr{R}_0 \cap \mathcal{PD} = \emptyset; \quad \mathscr{C} \cap \mathcal{PD} = \emptyset.$$

Other properties of distributions from the class $\mathcal{PD}$ are presented in [64].

Although Section 2.1 reviews results on the asymptotic behaviour of tail expectations of random and randomly weighted sums, a theorem from [97] (see Theorem 4 and the remark below this theorem) is recalled below due to its close relevance to the main results of this chapter.

**THEOREM 4.1.** *Let $\{\xi_1, \ldots, \xi_n\}$ be a collection of real-valued pQAI r.v.s such that $F_{\xi_L} \in \mathscr{C}$ for some $L \in \{1, \ldots, n\}$, and for other indices $k \neq L$, suppose $F_{\xi_k} \in \mathscr{C}$ or $\mathbb{P}\big(|\xi_k| > x\big) = o\big(\overline{F}_{\xi_L}(x)\big)$. If $\max\limits_{1\leqslant k\leqslant n} \mathbb{E}\big(\xi_k^+\big)^\alpha < \infty$, then for each $\beta \in [0, \alpha]$,*

$$\mathbb{E}\Big(\big(S_n^\xi\big)^\beta \mathbb{1}_{\left\{S_n^\xi > x\right\}}\Big) \sim \sum_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k^\beta \, \mathbb{1}_{\{\xi_k > x\}}\Big), \tag{4.1}$$

*and for each $\beta \in (0, \alpha]$,*

$$\mathbb{E}\left(\Big(\big(S_n^\xi - x\big)^+\Big)^\beta\right) \sim \sum_{k\in\mathcal{I}_n} \mathbb{E}\Big(\big(\xi_k - x\big)^+\Big)^\beta, \tag{4.2}$$

*where $\mathcal{I}_n \subseteq \{1, \ldots, n\}$ is a subset of indices such that $F_{\xi_k} \in \mathscr{C}$ for all $k \in \mathcal{I}_n$.*

The following discussion provides known results for the HG risk measure. Tang and Yang in [105] derived an explicit formula for calculating the HG risk measure when the Young function is a power function $\varphi(t) = t^{\varkappa}$, $\varkappa \geqslant 1$ (see Theorem 4.2 below). However, in the same article, authors remarked that, in the general setting, no closed-form expression for the HG risk measure is available, since its evaluation is related to the determination of quantiles. In such cases, Tang and Yang suggested to use asymptotic formulas. In particular, authors provided an asymptotic formula for random risks with regularly varying tails (see Theorem 4.3 below).

**THEOREM 4.2.** *Consider the power Young function $\varphi(t) = t^{\varkappa}$ with $\varkappa \geqslant 1$.*

(i) *If $q \in (0,1)$, $\varkappa = 1$, and $\mathbb{E}X^+ < \infty$, then*

$$HG_q(X) = F_X^{\leftarrow}(q) + \frac{\mathbb{E}\Big(X - F_X^{\leftarrow}(q)\Big)^+}{1-q}, \tag{4.3}$$

*where $F_X^{\leftarrow}(q) = \inf\big\{x \in \mathbb{R} : F_X(x) \geqslant q\big\} = \inf\{x \in \mathbb{R} : \overline{F}_X(x) \leqslant 1-q\}$ is the quantile function of r.v. $X$.*

(ii) *If $q \in (0,1)$, $\varkappa > 1$, $\mathbb{P}\big(X = F_X^{\leftarrow}(1)\big) = 0$, and $\mathbb{E}(X^+)^{\varkappa} < \infty$, then*

$$HG_q(X) = x + \left(\frac{\mathbb{E}\big((X-x)^+\big)^{\varkappa}}{1-q}\right)^{1/\varkappa}, \tag{4.4}$$

*where $x = x(q) \in \Big(-\infty, F_X^{\leftarrow}(1)\Big)$ is the unique solution of the equation*

$$\frac{\Big(\mathbb{E}\big((X-x)^+\big)^{\varkappa-1}\Big)^{\varkappa}}{\Big(\mathbb{E}\big((X-x)^+\big)^{\varkappa}\Big)^{\varkappa-1}} = 1-q. \tag{4.5}$$

**THEOREM 4.3.** *Consider the power Young function $\varphi(t) = t^{\varkappa}$ with $\varkappa \geqslant 1$, and let $X$ be a real-valued r.v. such that $F_X \in \mathscr{R}_\alpha$ for some $\alpha > \varkappa$. Then*

$$HG_q(X) \underset{q\uparrow 1}{\sim} \frac{\alpha(\alpha-\varkappa)^{\frac{\varkappa}{\alpha}-1}}{\varkappa^{\frac{\varkappa-1}{\alpha}}}\big(B(\alpha-\varkappa,\varkappa)\big)^{\frac{1}{\alpha}} F_X^{\leftarrow}(q),$$

*where*

$$B(a,b) = \int\limits_0^1 y^{a-1}(1-y)^{b-1}\mathrm{d}y, \quad a>0, \; b>0$$

*is the beta function.*

In this chapter, asymptotic formulas for the HG risk measure are derived that are similar to those given in Theorem 4.3, but for distributions from the broader class $\mathscr{C}$. More specifically, asymptotic expressions are obtained for the HG risk measure in the case where the risk variable $X$ represents a sum of consistently varying risks $S_n^{\xi}$. The main results of this paper are based on general formulas (4.4), (4.5), together with asymptotic relations (4.1), (4.2).

It should be noted that the derived asymptotic formulas are also applicable to the randomly weighted sum $S_n^{\theta\xi}$. The transition from asymptotic properties of the sum $S_n^{\xi}$ to analogous properties of the randomly weighted sum $S_n^{\theta\xi}$ is ensured by the product convolution properties reviewed in detail in Chapter 5 of [71]. Recall that, for two independent r.v.s $X$ and $Y$, the distribution of the product $XY$ is characterized by the product convolution of the d.f.s $F_X \otimes F_Y$. A particular result concerning closure properties under product convolution is formulated below (see Proposition 5.2(iii), Proposition 5.3(iii), Proposition 5.4(i) in [71]).

**THEOREM 4.4.** *Let $F$ be a distribution on $\mathbb{R}$, and let $G$ be a distribution such that $G(0-)=0$ and $G(0)<1$.*

(i) *If $F \in \mathscr{R}_\alpha$ and $\overline{G}(yx) = o\big(\overline{F}(x)\big)$ for some $y>0$, then $F \otimes G \in \mathscr{R}_\alpha$;*

(ii) *If $F \in \mathscr{C}$ and $\overline{G}(yx) = o\big(\overline{F}(x)\big)$ for some $y>0$, then $F \otimes G \in \mathscr{C}$;*

(iii) *If $F \in \mathscr{D}$, then $F \otimes G \in \mathscr{D}$.*

In actuarial contexts, randomly weighted sums are commonly used to model the present value of an insurer's aggregate future net losses over a finite horizon of $n$ time periods. Here, the real-valued r.v. $X_k$ acts as a net loss during the $k$-th period, while the corresponding random weight $\theta_k$ is a stochastic discount factor from time $k$ to the present time.

An analogous interpretation arises in financial risk management through portfolio modelling. A portfolio may be viewed as a collection of $n$ risk components, each characterised by a random loss $X_k$ and a random weight $\theta_k$ capturing the market environment. Under this framework, the sum $S_n^{\theta\xi}$ represents the aggregate future losses potentially incurred by the investment portfolio.

The problems related to the asymptotic behaviour of the sum $S_n^{\theta\xi}$ were considered in [20, 24, 47, 49, 55, 57, 76, 82, 102, 103, 109, 112, 132, 134, 136, 137], and asymptotic properties of distributions of sums $S_n^{\theta\xi}$ related to the asymptotic behaviour of risk measures, similar to the ES risk measure, were considered in [7, 42, 56, 107, 114, 139].

## 4.2 Main Results

The main results of this chapter are published in [92] and are presented below. These results consist of two theorems describing the asymptotic behaviour of the HG risk measure for random sums with summands whose distributions belong to the class $\mathscr{C}$. The first theorem is formulated for the sum of unweighted r.v.s, while the second theorem addresses sums of weighted r.v.s.

**THEOREM 4.5.** *Consider the power Young function $\varphi(t) = t^{\varkappa}$ with $\varkappa \geqslant 1$, and let $\{\xi_1, \dots, \xi_n\}$ be a collection of real-valued pQAI r.v.s.*

(i) *Let $\varkappa = 1$, $\max\limits_{1\leqslant k\leqslant n} \mathbb{E}\big(\xi_k^+\big) < \infty$, $F_{\xi_L} \in \mathscr{C}$ for some $L \in \{1, \dots, n\}$, and for other indices $k \neq L$, let $F_{\xi_k} \in \mathscr{C}$ or $\mathbb{P}\big(|\xi_k| > x\big) = o\big(\overline{F}_{\xi_L}(x)\big)$. If $F_{\xi_k} \in \mathcal{PD}$ for all $k \in \mathcal{I}_n = \big\{k \in \{1, \dots, n\} : F_{\xi_k} \in \mathscr{C}\big\}$, then*

$$HG_q\Big(S_n^{\xi}\Big) \underset{q\uparrow 1}{\sim} H_n^{\leftarrow}(q) + \frac{1}{1-q} \sum_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+,$$

*where $H_n^{\leftarrow}$ is the quantile function of the d.f.*

$$H_n := \max\Big\{0, 1 - \sum_{k\in\mathcal{I}_n} \overline{F}_{\xi_k}\Big\}.$$

(ii) *Let $\varkappa \geqslant 2$, $\max\limits_{1\leqslant k\leqslant n} \mathbb{E}\big(\xi_k^+\big)^{\varkappa} < \infty$, $F_{\xi_L} \in \mathscr{C}$ for some $L \in \{1, \dots, n\}$, and for other indices $k \neq L$, let $F_{\xi_k} \in \mathscr{C}$ or $\mathbb{P}\big(|\xi_k| > x\big) = o\big(\overline{F}_{\xi_L}(x)\big)$. Let, in addition,*

$$\left(\max_{k\in\mathcal{I}_n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(yx)}{\overline{F}_{\xi_k}(x)}\right)^{\varkappa}\left(\max_{k\in\mathcal{I}_n}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_k}(yx)}\right)^{\varkappa-1}<1 \tag{4.6}$$

*for some* $y>1$*, where* $\mathcal{I}_n=\big\{k\in\{1,\ldots,\}:F_{\xi_k}\in\mathscr{C}\big\}$*. Then*

$$HG_q\Big(S_n^{\xi}\Big)\underset{q\uparrow 1}{\sim}\widehat{x}(q)+\left(\frac{1}{1-q}\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa},$$

*where* $\widehat{x}=\widehat{x}(q)$ *is the unique solution to the equation*

$$\frac{\left(\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}\big)^+\Big)^{\varkappa-1}\right)^{\varkappa}}{\left(\sum_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}\big)^+\Big)^{\varkappa}\right)^{\varkappa-1}}=1-q.$$

As mentioned earlier, the second theorem addresses the asymptotic behaviour of the HG risk measure for weighted sums. In contrast to Theorem 4.5, this result assumes that distributions of all primary r.v.s, rather than only a subset, belong to the class $\mathscr{C}$.

**THEOREM 4.6.** *Consider the same power Young function as in Theorem 4.5. Let* $\{X_1,\ldots,X_n\}$ *be a collection of real-valued pQAI r.v.s with corresponding d.f.s* $\{F_{X_1},\ldots,F_{X_n}\}$ *belonging to the class* $\mathscr{C}$*. Let* $\{\theta_1,\ldots,\theta_n\}$ *be another collection of arbitrary dependent, nonnegative, and nondegenerate at zero r.v.s. Suppose that collections* $\{X_1,\ldots,X_n\}$ *and* $\{\theta_1,\ldots,\theta_n\}$ *are independent and that* $\max\limits_{1\leqslant k\leqslant n}\{\mathbb{E}\theta_k^p\}$ *is finite for some* $p>\max\limits_{1\leqslant k\leqslant n}J^+_{F_{X_k}}$.

(i) *If* $\varkappa=1$*,* $\max\limits_{1\leqslant k\leqslant n}\mathbb{E}\big(X_k^+\big)<\infty$*, and* $F_{X_k}\in\mathcal{PD}$ *for all* $k\in\{1,\ldots,n\}$*, then*

$$HG_q\Big(S_n^{\theta X}\Big)\underset{q\uparrow 1}{\sim}H_n^{*\leftarrow}(q)+\frac{1}{1-q}\sum_{k=1}^{n}\mathbb{E}\Big(\theta_kX_k-H_n^{*\leftarrow}(q)\Big)^+,$$

*where* $H_n^{*\leftarrow}$ *is the quantile function of the d.f.*

$$H_n^*:=\max\Big\{0,1-\sum_{k=1}^{n}\overline{F}_{\theta_kX_k}\Big\}.$$

(ii) *Let* $\varkappa \geqslant 2$ *be such that* $\max\limits_{1\leqslant k\leqslant n} \mathbb{E}\big(X_k^+\big)^{\varkappa} < \infty$. *Let, in addition,*

$$\left(\max_{1\leqslant k\leqslant n} \limsup_{x\to\infty} \frac{\overline{F}_{X_k}(yx)}{\overline{F}_{X_k}(x)}\right)^{\varkappa} \left(\max_{1\leqslant k\leqslant n} \limsup_{x\to\infty} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_k}(yx)}\right)^{\varkappa-1} < 1$$

*for some* $y > 1$. *Then*

$$HG_q\Big(S_n^{\theta X}\Big) \underset{q\uparrow 1}{\sim} x^*(q) + \left(\frac{1}{1-q}\sum_{k=1}^{n} \mathbb{E}\Big(\big(\theta_k X_k - x^*(q)\big)^+\Big)^{\varkappa}\right)^{1/\varkappa},$$

*where* $x^* = x^*(q)$ *is the unique solution to the equation*

$$\frac{\left(\sum_{k=1}^{n} \mathbb{E}\Big(\big(\theta_k X_k - x^*\big)^+\Big)^{\varkappa-1}\right)^{\varkappa}}{\left(\sum_{k=1}^{n} \mathbb{E}\Big(\big(\theta_k X_k - x^*\big)^+\Big)^{\varkappa}\right)^{\varkappa-1}} = 1 - q.$$

## 4.3 Proofs of the Main Results

### 4.3.1 Auxiliary Lemmas

This subsection provides auxiliary results concerning properties of quantile functions which are necessary for the proofs of the main results. Analogous statements to those formulated below can be found in [14, 15, 16, 17, 58, 65, 105, 116].

**LEMMA 4.1.** *Let* $F$ *be a d.f. from the class* $\mathscr{C}$. *Then*

$$F\Big(F^{\leftarrow}(q)\Big) \underset{q\uparrow 1}{\sim} q, \quad \text{and} \quad \overline{F}\Big(\overline{F}^{\leftarrow}(p)\Big) \underset{p\downarrow 0}{\sim} p,$$

*where*

$$F^{\leftarrow}(q) = \inf\big\{x : F(x) \geqslant q\big\},\ q \in (0,1),$$

*is the quantile function for* $F$, *and*

$$\overline{F}^{\leftarrow}(p) = \inf\big\{x : \overline{F}(x) \leqslant p\big\},\ p \in (0,1),$$

*is the quantile function for t.f.* $\overline{F}$.

*Proof.* Let $q \in (0,1)$. Since d.f. is right-continuous, the set

$$\{x : F(x) \geqslant q\} := \mathcal{A}_q$$

is closed. Hence, $F^{\leftarrow}(q) \in \mathcal{A}_q$, which implies that

$$F\Big(F^{\leftarrow}(q)\Big) \geqslant q \tag{4.7}$$

for all $q \in (0,1)$. Therefore

$$F\Big(F^{\leftarrow}(q)\Big) \underset{q\uparrow 1}{\gtrsim} q. \tag{4.8}$$

For the first asymptotic relation of Lemma 4.1, it remains to prove that

$$\limsup_{q\uparrow 1} \frac{F\Big(F^{\leftarrow}(q)\Big)}{q} \leqslant 1. \tag{4.9}$$

Suppose, on the contrary, that

$$\limsup_{q\uparrow 1} \frac{F\Big(F^{\leftarrow}(q)\Big)}{q} > 1.$$

Then, there exist $\Delta > 0$ and a sequence $q_n \uparrow 1$ such that

$$\lim_{n\to\infty} \frac{F\Big(F^{\leftarrow}(q_n)\Big)}{q_n} = 1 + \Delta,$$

implying that $F\big(F^{\leftarrow}(q_n)\big) > \big(1 + \frac{\Delta}{2}\big)q_n$ for large $n$. Since $F(x) \leqslant 1$ for all $x$, it holds that

$$q_n < \Big(1 + \frac{\Delta}{2}\Big)^{-1}$$

for large $n$. This is impossible because $q_n \uparrow 1$. The obtained contradiction proves estimate (4.9). Estimates (4.8) and (4.9) imply the first asymptotic relation of the lemma.

Now consider the second relation of Lemma 4.1 and let $p > 0$. Due to inequality (4.7), it holds that

$$\overline{F}\big(\overline{F}^{\leftarrow}(p)\big) = 1 - F\big(\inf\{x : \overline{F}(x) \leqslant p\}\big) = 1 - F\big(\inf\{x : F(x) \geqslant 1 - p\}\big)$$

$$= 1 - F\big(F^{\leftarrow}(1-p)\big) \leqslant p, \tag{4.10}$$

implying that

$$\overline{F}\Big(\overline{F}^{\leftarrow}(p)\Big) \underset{p\downarrow 0}{\lesssim} p. \tag{4.11}$$

It remains to prove that

$$\liminf_{p\downarrow 0} \frac{\overline{F}\big(\overline{F}^{\leftarrow}(p)\big)}{p} \geqslant 1. \tag{4.12}$$

Suppose, on the contrary, that

$$\liminf_{p\downarrow 0} \frac{\overline{F}\big(\overline{F}^{\leftarrow}(p)\big)}{p} < 1.$$

Then, there exist $\Delta \in (0,1)$ and a sequence $p_n \downarrow 0$ such that

$$\lim_{n\to\infty} \frac{\overline{F}\big(\overline{F}^{\leftarrow}(p_n)\big)}{p_n} = 1 - \Delta.$$

Hence,

$$\overline{F}\Big(\overline{F}^{\leftarrow}(p_n)\Big) = \overline{F}\big(\inf\{x : \overline{F}(x) \leqslant p_n\}\big) < \Big(1 - \frac{\Delta}{2}\Big)p_n,$$

or, equivalently,

$$F\big(\inf\{x : \overline{F}(x) \leqslant p_n\}\big) = F\big(\inf\{x : F(x) \geqslant 1 - p_n\big) > 1 - p_n + \frac{\Delta}{2}p_n$$

for large $n$.

For notational convenience, temporarily denote

$$x_n := \inf\{x : F(x) \geqslant 1 - p_n\}.$$

Since $\overline{F}(x) > 0$ for all $x$, the sequence $x_n$ is nondecreasing, and $x_n \uparrow \infty$. In addition, the d.f. $F$ is right-continuous, so that $F(x_n) \geqslant 1 - p_n$. For a fixed $n$, there are two possible cases: $F(x_n) = 1 - p_n$ or $F(x_n) > 1 - p_n$. In the first case, one can obtain

$$1 - p_n > 1 - p_n + \frac{\Delta}{2}p_n,$$

which is impossible due to the positivity of $\Delta$. Consequently, the d.f. $F$

has a jump at each point $x_n$, from which it follows that

$$F\big((1-\varepsilon_n)x_n\big) < 1-p_n \quad \text{and} \quad F(x_n) > (1-p_n) + \frac{\Delta}{2}p_n$$

or, equivalently,

$$\overline{F}\big((1-\varepsilon_n)x_n\big) > p_n \quad \text{and} \quad \overline{F}(x_n) < p_n\Big(1-\frac{\Delta}{2}\Big)$$

for some sequence $\varepsilon_n \downarrow 0$. These estimates imply that

$$\lim_{n\to\infty} \frac{\overline{F}\big((1-\varepsilon_n)x_n\big)}{\overline{F}(x_n)} \geqslant \frac{2}{2-\Delta} > 1$$

for some sequences $x_n \uparrow \infty$ and $\varepsilon_n \downarrow 0$. The derived relation contradicts the condition $F \in \mathscr{C}$, because the following should hold

$$\lim_{\varepsilon\downarrow 0} \limsup_{x\to\infty} \frac{\overline{F}\big((1-\varepsilon)x\big)}{\overline{F}(x)} \leqslant 1$$

by Definition 1.5.

The obtained contradiction proves inequality (4.12), which, together with estimate (4.11), prove the second asymptotic relation of Lemma 4.1. □

**LEMMA 4.2.** *Let $F$ be a d.f. from the class $\mathcal{PD}$. Then,*

$$\overline{F}^{\leftarrow}\big(\overline{F}(x)\big) \sim x, \quad \text{and} \quad F^{\leftarrow}\big(F(x)\big) \sim x,$$

*where the quantile functions $\overline{F}^{\leftarrow}$ and $F^{\leftarrow}$ are defined in Lemma 4.1.*

*Proof.* Consider the first asymptotic relation. By the definition of the quantile function $\overline{F}^{\leftarrow}$ it holds that

$$\overline{F}^{\leftarrow}\big(\overline{F}(x)\big) = \inf\{y : \overline{F}(y) \leqslant \overline{F}(x)\}.$$

Since the t. f. $\overline{F}$ is nonincreasing, meaning that $\overline{F}(y) \leqslant \overline{F}(x)$ for $y \geqslant x$, it holds that $\overline{F}^{\leftarrow}\big(\overline{F}(x)\big) \leqslant x$, implying

$$\overline{F}^{\leftarrow}\big(\overline{F}(x)\big) \lesssim x. \tag{4.13}$$

It remains to prove that

$$\overline{F}^{\leftarrow}\left(\overline{F}(x)\right) \gtrsim x. \tag{4.14}$$

Suppose, on the contrary, that

$$\liminf_{x\to\infty} \frac{\overline{F}^{\leftarrow}\left(\overline{F}(x)\right)}{x} < 1.$$

In such a case, there exist $\Delta \in (0,1)$ and a sequence $x_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \frac{\overline{F}^{\leftarrow}\left(\overline{F}(x_n)\right)}{x_n} = 1 - \Delta.$$

Hence,

$$\overline{F}^{\leftarrow}\left(\overline{F}(x_n)\right) = \inf\{y : \overline{F}(y) \leqslant \overline{F}(x_n)\} \leqslant \left(1 - \frac{2\Delta}{3}\right)x_n$$

for sufficiently large $n$, say $n \geqslant N_\Delta$. This relation implies that

$$\overline{F}\left(\left(1 - \frac{\Delta}{2}\right)x_n\right) \leqslant \overline{F}(x_n).$$

Since the t. f. $\overline{F}$ is nonincreasing, it must hold

$$\overline{F}\left(\left(1 - \frac{\Delta}{2}\right)x_n\right) = \overline{F}(x_n)$$

for $n \geqslant N_\Delta$. Therefore,

$$\lim_{n\to\infty} \frac{\overline{F}\left(\left(1 - \frac{\Delta}{2}\right)x_n\right)}{\overline{F}(x_n)} = 1,$$

which contradicts Definition 4.1 of $\mathcal{PD}$ distributions class. Consequently, asymptotic relation (4.14) holds. Asymptotic bounds (4.13) and (4.14) imply the first relation of Lemma 4.2.

The second asymptotic relation of Lemma 4.2 follows from the equalities

$$\lim_{x\to\infty} \frac{F^{\leftarrow}\left(F(x)\right)}{x} = \lim_{x\to\infty} \frac{1}{x} \inf\{y : F(y) \geqslant F(x)\}$$

$$= \lim_{x\to\infty} \frac{1}{x} \inf\{y : \overline{F}(y) \leqslant \overline{F}(x)\} = \lim_{x\to\infty} \frac{\overline{F}^{\leftarrow}\left(\overline{F}(x)\right)}{x} = 1.$$

This ends the proof of Lemma 4.2. $\square$

**LEMMA 4.3.** *Let d. f.* $F \in \mathcal{PD}$. *Then, for any two positive functions* $a(x)$ *and* $b(x)$ *such that* $a(x) \sim b(x)$ *and* $a(x) \underset{x\to\infty}{\to} 0$, *it holds that*

$$\overline{F}^{\leftarrow}\big(a(x)\big) \sim \overline{F}^{\leftarrow}\big(b(x)\big)$$

*and, for any positive functions* $\hat{a}(p)$ *and* $\hat{b}(p)$ *such that* $\hat{a}(p) \underset{p\downarrow 0}{\sim} \hat{b}(p)$ *and* $\hat{a}(p) \underset{p\downarrow 0}{\to} 0$, *the following holds*

$$\overline{F}^{\leftarrow}\big(\hat{a}(p)\big) \underset{p\downarrow 0}{\sim} \overline{F}^{\leftarrow}\big(\hat{b}(p)\big).$$

*Proof.* It is sufficient to prove the first equivalence relation because the second relation follows from the first one by noting that

$$\lim_{p\downarrow 0} \frac{\overline{F}^{\leftarrow}\big(\hat{a}(p)\big)}{\overline{F}^{\leftarrow}\big(\hat{b}(p)\big)} = \lim_{x\to\infty} \frac{\overline{F}^{\leftarrow}\left(\hat{a}\left(\frac{1}{x}\right)\right)}{\overline{F}^{\leftarrow}\left(\hat{b}\left(\frac{1}{x}\right)\right)}.$$

Consider the first relation. Firstly, let's show that

$$\liminf_{x\to\infty} \frac{\overline{F}^{\leftarrow}\,(a(x))}{\overline{F}^{\leftarrow}\,(b(x))} \geqslant 1.$$

Assume that the opposite holds

$$\liminf_{x\to\infty} \frac{\overline{F}^{\leftarrow}\,(a(x))}{\overline{F}^{\leftarrow}\,(b(x))} < 1.$$

This means that there exist $\Delta > 0$ and a sequence $x_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \frac{\overline{F}^{\leftarrow}\,(a(x_n))}{\overline{F}^{\leftarrow}\,(b(x_n))} = 1 - \Delta.$$

It follows that

$$\frac{\overline{F}^{\leftarrow}\,(a(x_n))}{1-\frac{\Delta}{2}} < \overline{F}^{\leftarrow}\,(b(x_n))$$

for large enough $n$. By the definition of $\overline{F}^{\leftarrow}$, it holds that

$$\overline{F}\left(\frac{\overline{F}^{\leftarrow}\big(a(x_n)\big)}{1-\frac{\Delta}{2}}\right) > b(x_n).$$

This relation, together with (4.10), implies

$$\frac{\overline{F}\left(\left(1-\frac{\Delta}{2}\right)\frac{\overline{F}^{\leftarrow}(a(x_n))}{1-\frac{\Delta}{2}}\right)}{\overline{F}\left(\frac{\overline{F}^{\leftarrow}(a(x_n))}{1-\frac{\Delta}{2}}\right)} = \frac{\overline{F}\left(\overline{F}^{\leftarrow}\big(a(x_n)\big)\right)}{\overline{F}\left(\frac{\overline{F}^{\leftarrow}(a(x_n))}{1-\frac{\Delta}{2}}\right)} \leqslant \frac{a(x_n)}{b(x_n)} \underset{n\to\infty}{\to} 1,$$

since $a(x) \sim b(x)$. Thus,

$$\lim_{n\to\infty} \frac{\overline{F}\left(\left(1-\frac{\Delta}{2}\right)\frac{\overline{F}^{\leftarrow}(a(x_n))}{1-\frac{\Delta}{2}}\right)}{\overline{F}\left(\frac{\overline{F}^{\leftarrow}(a(x_n))}{1-\frac{\Delta}{2}}\right)} \leqslant 1$$

for some sequence $\dfrac{\overline{F}^{\leftarrow}\big(a(x_n)\big)}{1-\frac{\Delta}{2}} \underset{n\to\infty}{\to} \infty$. This contradicts Definition 4.1.

It remains to prove that

$$\limsup_{x\to\infty} \frac{\overline{F}^{\leftarrow}(a(x))}{\overline{F}^{\leftarrow}(b(x))} \leqslant 1.$$

Suppose, on the contrary, that

$$\limsup_{x\to\infty} \frac{\overline{F}^{\leftarrow}(a(x))}{\overline{F}^{\leftarrow}(b(x))} > 1.$$

It follows that there exist $\Delta > 0$ and a sequence $x_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \frac{\overline{F}^{\leftarrow}(a(x_n))}{\overline{F}^{\leftarrow}(b(x_n))} = 1 + \Delta.$$

Consequently,

$$\overline{F}^{\leftarrow}(a(x_n)) > \Big(1+\frac{\Delta}{2}\Big)\overline{F}^{\leftarrow}(b(x_n))$$

for large $n$. By the definition of $\overline{F}^{\leftarrow}$ it follows that

$$a(x_n) < \overline{F}\left(\Big(1+\frac{\Delta}{2}\Big)\overline{F}^{\leftarrow}\big(b(x_n)\big)\right).$$

This relation, together with (4.10), implies

$$\frac{\overline{F}\Big(\Big(1+\frac{\Delta}{2}\Big)\overline{F}^{\leftarrow}\big(b(x_n)\big)\Big)}{\overline{F}\Big(\overline{F}^{\leftarrow}\big(b(x_n)\big)\Big)} \geqslant \frac{a(x_n)}{b(x_n)} \underset{n\to\infty}{\to} 1,$$

since $a(x) \sim b(x)$. Thus,

$$\lim_{n\to\infty} \frac{\overline{F}\Big(\Big(1+\frac{\Delta}{2}\Big)\overline{F}^{\leftarrow}\big(b(x_n)\big)\Big)}{\overline{F}\Big(\overline{F}^{\leftarrow}\big(b(x_n)\big)\Big)} \geqslant 1$$

for some sequence $\overline{F}^{\leftarrow}\big(b(x_n)\big) \underset{n\to\infty}{\to} \infty$. This contradicts Definition 4.1. This ends the proof of Lemma 4.3. □

**LEMMA 4.4.** *Let $F$ and $G$ be d.f.s such that $F \in \mathscr{C} \cap \mathcal{PD}$ and $\overline{G}(x) \sim c\overline{F}(x)$ for some constant $c > 0$. Then,*

$$G^{\leftarrow}(1-p) \underset{p\downarrow 0}{\sim} F^{\leftarrow}\Big(1-\frac{p}{c}\Big).$$

*Proof.* The condition $F \in \mathcal{PD}$ and Lemmas 4.2 and 4.3 imply that

$$x \sim \overline{F}^{\leftarrow}\big(\overline{F}(x)\big) \sim \overline{F}^{\leftarrow}\left(\frac{\overline{G}(x)}{c}\right). \tag{4.15}$$

The conditions $F \in \mathscr{C} \cap \mathcal{PD}$ and $\overline{G}(x) \sim c\overline{F}(x)$ imply that $G \in \mathscr{C} \cap \mathcal{PD}$. Hence, $\overline{G}(x) > 0$ for all $x \in \mathbb{R}$, and, therefore, $\overline{G}^{\leftarrow}(p) \to \infty$ as $p \downarrow 0$. By taking $x = \overline{G}^{\leftarrow}(p)$ in (4.15), one can obtain

$$\overline{G}^{\leftarrow}(p) \underset{p\downarrow 0}{\sim} \overline{F}^{\leftarrow}\bigg(\frac{1}{c}\overline{G}\Big(\overline{G}^{\leftarrow}(p)\Big)\bigg).$$

By Lemma 4.1, the following holds

$$\overline{G}\Big(\overline{G}^{\leftarrow}(p)\Big) \underset{p\downarrow 0}{\sim} p.$$

Hence, from Lemma 4.3, one can derive

$$\overline{G}^{\leftarrow}(p) \underset{p\downarrow 0}{\sim} \overline{F}^{\leftarrow}\Big(\frac{p}{c}\Big)$$

or, equivalently,

$$G^{\leftarrow}(1-p) \underset{p\downarrow 0}{\sim} F^{\leftarrow}\Big(1-\frac{p}{c}\Big).$$

This ends the proof of Lemma 4.4. $\square$

The following discussion examines several properties of the special function defined in equation (4.5).

Let $X$ be a r.v. such that $\mathbb{E}(X^+)^{\varkappa} < \infty$ and $\overline{F}_X(x) = \mathbb{P}(X > x) > 0$ for all $x \in \mathbb{R}$. Let

$$\gamma_X(x) := \frac{\Big(\mathbb{E}\big((X-x)^+\big)^{\varkappa-1}\Big)^{\varkappa}}{\Big(\mathbb{E}\big((X-x)^+\big)^{\varkappa}\Big)^{\varkappa-1}}, \quad x \in \mathbb{R}, \tag{4.16}$$

be a well-defined function from equation (4.5). The following five lemmas provide properties of the function $\gamma_X$ and its components.

**LEMMA 4.5.** *Let $X$ be a r.v. such that $\mathbb{E}(X^+)^{\varkappa} < \infty$ and $\overline{F}_X(x) > 0$ for all $x \in \mathbb{R}$. If $\varkappa > 1$, then function $g(x) := \mathbb{E}\big((X-x)^+\big)^{\varkappa}$ is continuously differentiable with*

$$g'(x) = -\varkappa\mathbb{E}\big((X-x)^+\big)^{\varkappa-1}, \quad x \in \mathbb{R}.$$

*If $\varkappa = 1$, then for each $x \in \mathbb{R}$ the derivative from the right $g'_+(x) = -\overline{F}(x)$ and the derivative from the left $g'_-(x) = -\overline{F}(x-0)$.*

For the proof of the above result see Lemma 2.1 in [105].

The following statements address properties of the function $\gamma_X$.

**LEMMA 4.6.** *Let $X$ be a r.v. such that $\mathbb{E}(X^+)^{\varkappa} < \infty$ and $\overline{F}_X(x) > 0$ for all $x \in \mathbb{R}$. Let $\gamma_X$ be a function defined in equation (4.16). If $\varkappa > 1$, then*

$$\lim_{x\to-\infty} \gamma_X(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} \gamma_X(x) = 0.$$

*Proof.* While [105] outlines the main steps of the proof (see the proof of Theorem 2.1), a full and more in-depth proof is presented here for completeness.

Since $F_X(x) > 0$, it holds that

$$\mathbb{E}\big((X-x)^+\big)^{\varkappa} = \varkappa \int\limits_x^{\infty} (u-x)^{\varkappa-1}\overline{F}_X(u)\mathrm{d}u > 0$$

for any $x \in \mathbb{R}$. By the classical Hölder's inequality,

$$\gamma_X(x) = \frac{\Big(\mathbb{E}\Big(\big((X-x)^+\big)^{\varkappa-1}\mathbb{1}_{(x,\infty)}(X)\Big)\Big)^{\varkappa}}{\big(\mathbb{E}\big((X-x)^+\big)^{\varkappa}\big)^{\varkappa-1}} \leqslant \frac{1}{\big(\mathbb{E}\big((X-x)^+\big)^{\varkappa}\big)^{\varkappa-1}}$$

$$\times\left\{\Big(\mathbb{E}\big((X-x)^+\big)^{(\varkappa-1)\frac{\varkappa}{\varkappa-1}}\Big)^{\frac{\varkappa-1}{\varkappa}}\Big(\mathbb{E}\Big(\mathbb{1}_{(x,\infty)}(X)\Big)^{\varkappa}\Big)^{\frac{1}{\varkappa}}\right\}^{\varkappa} = \overline{F}_X(x),$$

which implies the second equality of Lemma 4.6.

Consider the first equality of Lemma 4.6. It is obvious that

$$\lim_{x\to-\infty}\gamma_X(x) = \lim_{x\to\infty}\frac{\left(\mathbb{E}\left(\frac{(X+x)^+}{x}\right)^{\varkappa-1}\right)^{\varkappa}}{\left(\mathbb{E}\left(\frac{(X+x)^+}{x}\right)^{\varkappa}\right)^{\varkappa-1}}. \tag{4.17}$$

The classical $c_r$-inequality and inequality $(a+b)^+ \leqslant a^+ + b^+$ provided for any $a, b \in \mathbb{R}$ imply that, for positive $x$,

$$\mathbb{E}\left(\frac{(X+x)^+}{x}\right)^{\varkappa-1} \leqslant \mathbb{E}\left(\frac{X^+ + x}{x}\right)^{\varkappa-1}$$

$$\leqslant c_{\varkappa}\left(\frac{1}{x^{\varkappa-1}}\mathbb{E}\big(X^+\big)^{\varkappa-1} + 1\right) < \infty,$$

where

$$c_{\varkappa} = \begin{cases} 1 & \text{if} \quad 1 < \varkappa \leqslant 2, \\ 2^{\varkappa-2} & \text{if} \quad \varkappa > 2. \end{cases}$$

Hence,

$$\lim_{x\to\infty}\left(\mathbb{E}\left(\frac{(X+x)^+}{x}\right)^{\varkappa-1}\right)^{\varkappa} = \left(\mathbb{E}\lim_{x\to\infty}\left(\frac{(X+x)^+}{x}\right)^{\varkappa-1}\right)^{\varkappa} = 1$$

by the dominated convergence theorem.

Similarly,

$$\lim_{x\to\infty}\left(\mathbb{E}\left(\frac{(X+x)^+}{x}\right)^{\varkappa}\right)^{\varkappa-1}=1.$$

Now the first equality of Lemma 4.6 can be derived by substituting the last two relations into equality (4.17).

This ends the proof of Lemma 4.6. $\square$

**LEMMA 4.7.** *Let $X$ be a r.v. such that $\mathbb{E}\big(X^+\big)^{\varkappa}<\infty$ and $\overline{F}_X(x)>0$ for $x\in\mathbb{R}$. If $\varkappa\geqslant 2$, then the function $\gamma_X$ is continuous and strongly decreasing on $\mathbb{R}$.*

*Proof.* Firstly, consider the case $\varkappa>2$. By denoting $\eta_x:=(X-x)^+$,

$$\gamma_X(x)=\frac{\big(\mathbb{E}\eta_x^{\varkappa-1}\big)^{\varkappa}}{\big(\mathbb{E}\eta_x^{\varkappa}\big)^{\varkappa-1}}.$$

Due to Lemma 4.5,

$$\gamma_X'(x)=-\varkappa(\varkappa-1)\frac{\big(\mathbb{E}\eta_x^{\varkappa-1}\big)^{\varkappa-1}}{\big(\mathbb{E}\eta_x^{\varkappa}\big)^{\varkappa}}\Big(\mathbb{E}\eta_x^{\varkappa-2}\mathbb{E}\eta_x^{\varkappa}-\big(\mathbb{E}\eta_x^{\varkappa-1}\big)^2\Big),\quad x\in\mathbb{R}.$$

The classical Cauchy-Schwarz inequality and condition $\overline{F}_X(x)>0$, $x\in\mathbb{R}$, imply that

$$\mathbb{E}\eta_x^{\varkappa-1}=\mathbb{E}\eta_x^{\frac{\varkappa-2}{2}}\eta_x^{\frac{\varkappa}{2}}<\big(\mathbb{E}\eta_x^{\varkappa-2}\big)^{\frac{1}{2}}\big(\mathbb{E}\eta_x^{\varkappa}\big)^{\frac{1}{2}}.$$

Therefore, $\gamma_X'(x)<0$ for $x\in\mathbb{R}$, implying the assertion of Lemma 4.7 in the case $\varkappa>2$.

Now suppose that $\varkappa=2$. In this case,

$$\gamma_X(x)=\frac{\big(\mathbb{E}\eta_x\big)^2}{\mathbb{E}\eta_x^2}.$$

Using Lemma 4.5, one can obtain that the derivative from the right

$$\big\{\gamma_X\big\}_+'(x)=-2\frac{\mathbb{E}\eta_x}{\big(\mathbb{E}\eta_x^2\big)^2}\Big(\overline{F}(x)\mathbb{E}\eta_x^2-\big(\mathbb{E}\eta_x\big)^2\Big)$$

is negative for any $x \in \mathbb{R}$ due to the Hölder inequality

$$\mathbb{E}\eta_x = \mathbb{E}\eta_x \mathbb{1}_{(x,\infty)}(X) < \left(\mathbb{E}\eta_x^2\right)^{\frac{1}{2}} \left(\mathbb{E}\mathbb{1}_{(x,\infty)}(X)\right)^{\frac{1}{2}} = \left(\mathbb{E}\eta_x^2\right)^{\frac{1}{2}} \left(\overline{F}(x)\right)^{\frac{1}{2}},$$

where the strictness of the inequality follows from the condition $\overline{F}_X(x) > 0,\ x \in \mathbb{R}$.

In the same way, one can get that the derivative from the left $\left\{\gamma_X\right\}_-'(x)$ is also negative for any $x \in \mathbb{R}$ because

$$\mathbb{E}\eta_x \mathbb{1}_{(x,\infty)}(X) = \mathbb{E}\eta_x \mathbb{1}_{[x,\infty)}(X),\ x \in \mathbb{R}.$$

The derived properties of the function $\gamma_X$ imply that this function is continuous and strongly decreasing on $\mathbb{R}$.

This ends the proof of Lemma 4.7. □

**LEMMA 4.8.** *Let $X$ be a r.v. such that $\mathbb{E}\left(X^+\right)^{\varkappa} < \infty$ and $F_X \in \mathscr{C}$. If $\varkappa > 1$, then*

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\gamma_X(xy)}{\gamma_X(x)} \leqslant 1.$$

.

*Proof.* If $\mathbb{E}\left(X^+\right)^{\varkappa}$ is finite, then

$$\mathbb{E}\left((X-x)^+\right)^p = p\int_x^\infty (u-x)^{p-1}\mathbb{P}(X>u)\mathrm{d}u$$

for any $0 < p \leqslant \varkappa$. Therefore, for all positive $x$ and $y$

$$\frac{\gamma_X(xy)}{\gamma_X(x)} = \left(\frac{\mathbb{E}\left((X-xy)^+\right)^{\varkappa-1}}{\mathbb{E}\left((X-x)^+\right)^{\varkappa-1}}\right)^{\varkappa} \left(\frac{\mathbb{E}\left((X-x)^+\right)^{\varkappa}}{\mathbb{E}\left((X-yx)^+\right)^{\varkappa}}\right)^{\varkappa-1}$$

$$= \left(\frac{\int_{xy}^\infty (u-xy)^{\varkappa-2}\overline{F}_X(u)\mathrm{d}u}{\int_x^\infty (u-x)^{\varkappa-2}\overline{F}_X(u)\mathrm{d}u}\right)^{\varkappa} \left(\frac{\int_x^\infty (u-x)^{\varkappa-1}\overline{F}_X(u)\mathrm{d}u}{\int_{xy}^\infty (u-xy)^{\varkappa-1}\overline{F}_X(u)\mathrm{d}u}\right)^{\varkappa-1}$$

$$= \left(\frac{\int_x^\infty (u-x)^{\varkappa-2}\overline{F}_X(uy)\mathrm{d}u}{\int_x^\infty (u-x)^{\varkappa-2}\overline{F}_X(u)\mathrm{d}u}\right)^{\varkappa} \left(\frac{\int_x^\infty (u-x)^{\varkappa-1}\overline{F}_X(u)\mathrm{d}u}{\int_x^\infty (u-x)^{\varkappa-1}\overline{F}_X(uy)\mathrm{d}u}\right)^{\varkappa-1}$$

$$\leqslant \left(\sup_{u\geqslant x} \frac{\overline{F}_X(uy)}{\overline{F}_X(u)}\right)^{\varkappa} \left(\sup_{u\geqslant x} \frac{\overline{F}_X(u)}{\overline{F}_X(uy)}\right)^{\varkappa-1}. \tag{4.18}$$

If $y \in (0,1)$, then

$$\limsup_{x\to\infty} \frac{\gamma_X(xy)}{\gamma_X(x)} \leqslant \left( \limsup_{x\to\infty} \sup_{u\geqslant x} \frac{\overline{F}_X(uy)}{\overline{F}_X(u)} \right)^{\varkappa}.$$

Since d.f. $F_X \in \mathscr{C}$ the last inequality implies the statement of Lemma 4.8. $\square$

**LEMMA 4.9.** *Let $X$ be a r.v. with d.f. $F_X$ such that $\mathbb{E}\big(X^+\big)^{\varkappa} < \infty$ and*

$$\left( \limsup_{x\to\infty} \frac{\overline{F}_X(yx)}{\overline{F}_X(x)} \right)^{\varkappa} \left( \limsup_{x\to\infty} \frac{\overline{F}_X(x)}{\overline{F}_X(yx)} \right)^{\varkappa-1} < 1 \tag{4.19}$$

*for some $\varkappa > 1$ and some $y > 1$. Then*

$$\limsup_{x\to\infty} \frac{\gamma_X(xy)}{\gamma_X(x)} < 1$$

*for some $y > 1$.*

**REMARK 4.1.** *Since $\overline{F}_X(x) \geqslant \overline{F}_X(vx)$ for all $v > 1$, condition* (4.19) *implies that $F_X \in \mathcal{PD}$.*

*If $F_X \in \mathscr{R}_\alpha$ for some $\alpha > 0$, then condition* (4.19) *holds for all $\varkappa > 1$.*

*If $F_X \in \mathscr{ERV}_{\{\alpha,\beta\}} \subset \mathscr{C}$ for some $0 < \alpha < \beta < \infty$, then condition* (4.19) *holds in the case $1 < \varkappa < \frac{\beta}{\beta-\alpha}$.*

*Proof.* Let $\varkappa > 1$ and $y > 1$ be such that condition (4.19) holds. By estimate (4.18),

$$\limsup_{x\to\infty} \frac{\gamma_X(xy)}{\gamma_X(x)} \leqslant \limsup_{x\to\infty} \left( \sup_{u\geqslant x} \frac{\overline{F}_X(uy)}{\overline{F}_X(u)} \right)^{\varkappa} \left( \sup_{u\geqslant x} \frac{\overline{F}_X(u)}{\overline{F}_X(uy)} \right)^{\varkappa-1}$$

$$= \left( \limsup_{x\to\infty} \frac{\overline{F}_X(xy)}{\overline{F}_X(x)} \right)^{\varkappa} \left( \limsup_{x\to\infty} \frac{\overline{F}_X(x)}{\overline{F}_X(xy)} \right)^{\varkappa-1}.$$

Now Lemma 4.9 follows from condition (4.19). $\square$

**REMARK 4.2.** *Let $\varkappa \geqslant 2$, and let $X$ be a r.v. with d.f. $F_X$ satisfying condition* (4.19). *Then, the function $\gamma_X$, defined in equality* (4.16), *is the t.f. of d.f. $1 - \gamma_X$ belonging to the class $\mathscr{C} \cap \mathcal{PD}$.*

*It immediately follows from Lemmas 4.4-4.9.*

The next four lemmas provide some closure properties of d.f.s with respect to the dependence structure and the product convolution. The result of these lemmas is used to derive Theorem 4.6.

**LEMMA 4.10.** *Let $X_1$, $X_2$ be QAI r.v.s with d.f. $F_{X_1} \in \mathscr{D}$, $F_{X_2} \in \mathscr{D}$. Let $\theta_1$, $\theta_2$ be two nonnegative, nondegenerate at zero r.v.s such that vectors $(X_1, X_2)$ and $(\theta_1, \theta_2)$ are independent. If $\max\{\theta_1^p, \theta_2^p\} < \infty$ for some $p > \max\{J^+_{F_{X_1}}, J^+_{F_{X_2}}\}$, then r.v.s $\theta_1 X_1$ and $\theta_2 X_2$ are QAI.*

For the proof of the above result see Lemma 4 in [58].

**LEMMA 4.11.** *Let $X$ and $\theta$ be independent r.v.s, and let d.f. $F_X$ belong to the class $\mathscr{C}$. If $\theta$ is a nonnegative, nondegenerate at zero r.v. such that $\mathbb{E}\theta^p < \infty$ for some $p > J^+_{F_X}$, then d.f. $F_{\theta X}$ of the product $\theta X$ also belongs to the class $\mathscr{C}$.*

The proof of the above result on the closure of class $\mathscr{C}$ with respect to the product convolution can be found in [116] (see Lemma 2.5).

**LEMMA 4.12.** *Let $X$ be a r.v. with d.f. $F_X \in \mathcal{PD} \cap \mathscr{D}$, and let $\theta$ be an independent of $X$, nonnegative, and nondegenerate at zero r.v. such that $\mathbb{E}\theta^p < \infty$ for some $p > J^+_{F_X}$. Then the product d.f. $F_{\theta X}$ belongs to the class $\mathcal{PD}$.*

*Proof.* Fix $y > 1$ such that

$$\limsup_{x\to\infty} \frac{\overline{F}_X(xy)}{\overline{F}_X(x)} < 1,$$

and let $a \in (0,1)$ and $b > 0$ be such that $\mathbb{P}(b > 0) > 0$. If $x$ is sufficiently large, then

$$\begin{aligned}
\frac{\overline{F}_{\theta X}(xy)}{\overline{F}_{\theta X}(x)} &= \frac{\mathbb{P}(\theta X > xy, \theta \leqslant x^a)}{\mathbb{P}(\theta X > x)} + \frac{\mathbb{P}(\theta X > xy, \theta > x^a)}{\mathbb{P}(\theta X > x)} \\
&\leqslant \frac{\mathbb{P}(\theta X > xy, \theta \leqslant x^a)}{\mathbb{P}(\theta X > x, \theta \leqslant x^a)} + \frac{\mathbb{P}(\theta > x^a)}{\mathbb{P}(\theta X > x, \theta > b)} \\
&\leqslant \frac{\mathbb{P}(\theta X > xy, 0 < \theta \leqslant x^a)}{\mathbb{P}(\theta X > x, 0 < \theta \leqslant x^a)} + \frac{\mathbb{E}\theta^p}{x^{ap}\mathbb{P}(\theta > b)\mathbb{P}(X > \frac{x}{b})} \\
&\leqslant \sup_{0<u\leqslant x^a} \frac{\overline{F}_X\left(\frac{yx}{u}\right)}{\overline{F}_X\left(\frac{x}{u}\right)} + \frac{\mathbb{E}\theta^p}{\mathbb{P}(\theta > b)} \frac{1}{x^{ap}\overline{F}_X(x)} \frac{\overline{F}_X(x)}{\overline{F}\left(\frac{x}{b}\right)}
\end{aligned}$$

$$= \sup_{z \geqslant x^{1-a}} \frac{\overline{F}_X(zy)}{\overline{F}_X(z)} + \frac{\mathbb{E}\theta^p}{\mathbb{P}(\theta > b)} \frac{1}{x^{ap}\overline{F}_X(x)} \frac{\overline{F}_X(x)}{\overline{F}\left(\frac{x}{b}\right)}. \tag{4.20}$$

Since $F_X \in \mathscr{D}$, it is obvious that $\overline{F}_X = O\big(\overline{F}_X(\frac{x}{b})\big)$. Moreover,

$$\lim_{x\to\infty} x^q \overline{F}_X(x) = \infty$$

for $q > J^+_{F_X}$ (see, e.g., Lemma 3.5 in [102]). If $p > J^+_{F_X}$, then there exists $a \in (0,1)$ such that $ap > J^+_{F_X}$. For this particular $a$, the second term on the right side of (4.20) tends to zero as $x$ tends to infinity. Consequently, for this particular $a \in (0,1)$,

$$\limsup_{x\to\infty} \frac{\overline{F}_{\theta X}(xy)}{\overline{F}_{\theta X}(x)} \leqslant \limsup_{x\to\infty} \sup_{z \geqslant x^{1-a}} \frac{\overline{F}_X(zy)}{\overline{F}_X(z)}$$

$$= \limsup_{x\to\infty} \frac{\overline{F}_X(xy)}{\overline{F}_X(x)} < 1. \tag{4.21}$$

Lemma 4.12 is proved. Note that a similar statement, but for a positive random weight, is proved in Theorem 4.1 of [65]. $\square$

**LEMMA 4.13.** *Let $X$ be a r.v. with d.f. $F_X \in \mathscr{D}$, and let $\theta$ be an independent of $X$, nonnegative, and nondegenerate at zero r.v. such that $\mathbb{E}\theta^p < \infty$ for some $p > J^+_{F_X}$. Then, for all $y > 0$,*

$$\limsup_{x\to\infty} \frac{\overline{F}_{\theta X}(yx)}{\overline{F}_{\theta X}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_X(yx)}{\overline{F}_X(x)}$$

*and*

$$\limsup_{x\to\infty} \frac{\overline{F}_{\theta X}(x)}{\overline{F}_{\theta X}(yx)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_X(x)}{\overline{F}_X(yx)}.$$

*Proof.* According to (4.21),

$$\limsup_{x\to\infty} \frac{\overline{F}_{\theta X}(yx)}{\overline{F}_{\theta X}(x)} \leqslant \limsup_{x\to\infty} \sup_{z \geqslant x^{1-a}} \frac{\overline{F}_X(yz)}{\overline{F}_X(z)} = \limsup_{x\to\infty} \frac{\overline{F}_X(yx)}{\overline{F}_X(x)}$$

with some special $a \in (0,1)$. Therefore, the first inequality of Lemma 4.13 holds.

Similarly to (4.20), one can obtain

$$\frac{\overline{F}_{\theta X}(x)}{\overline{F}_{\theta X}(yx)} \leqslant \sup_{z\geqslant x^{1-a}} \frac{\overline{F}(z)}{\overline{F}(yz)} + \frac{\mathbb{E}\theta^p}{\mathbb{P}(\theta > b)} \frac{1}{(yx)^{ap}\overline{F}_X(x)} \frac{\overline{F}_X(x)}{\overline{F}_X\left(\frac{yx}{b}\right)}$$

for large $x$, $b > 0$ such that $\mathbb{P}(\theta > b) > 0$ and for $a \in (0,1)$ such that $ap > J^+_{F_X}$. The same arguments as in the derivation of (4.21) imply the second inequality of Lemma 4.13.

This ends the proof of Lemma 4.13. □

### 4.3.2 Proof of Theorem 4.5

Consider the first part (i) of Theorem 4.5. By relation (4.1) of Theorem 4.1, it holds that

$$\mathbb{E}\Big(\big(S_n^\xi\big)^\beta \mathbb{1}_{\left\{S_n^\xi > x\right\}}\Big) \sim \sum_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k^\beta \mathbb{1}_{\{\xi_k > x\}}\Big), \tag{4.22}$$

for all $\beta \in [0,1]$. In the particular case $\beta = 0$,

$$\overline{F}_{S_n^\xi}(x) = \mathbb{P}\big(S_n^\xi > x\big) \sim \sum_{k\in\mathcal{I}_n} \overline{F}_{\xi_k}(x) \sim \overline{H}_n(x). \tag{4.23}$$

Notice that $\overline{F}_{\xi_k}(x) > 0$ for all $k \in \mathcal{I}_n$ if $x$ is sufficiently large. Hence, according to the min-max inequality (see Lemma 2.1),

$$\min_{k\in\mathcal{I}_n} \frac{\overline{F}_{\xi_k}(xy)}{\overline{F}_{\xi_k}(x)} \leqslant \frac{\overline{H}_n(yx)}{\overline{H}_n(x)} \leqslant \max_{k\in\mathcal{I}_n} \frac{\overline{F}_{\xi_k}(xy)}{\overline{F}_{\xi_k}(x)} \tag{4.24}$$

for each $y \in (0,1]$ and sufficiently large $x$. This double inequality shows that the d.f. $H_n$, together with the d.f. $F_{S_n^\xi}$, belongs to the class $\mathscr{C}\cap\mathcal{PD}$. Hence, by Lemma 4.4,

$$F^{\leftarrow}_{S_n^\xi}(q) \underset{q\uparrow 1}{\sim} H_n^{\leftarrow}(q). \tag{4.25}$$

For $q \in (0,1)$, by equality (4.3) and the min-max inequality,

$$\frac{HG_q\Big(S_n^\xi\Big)}{H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+}$$

$$\leqslant \max\left\{\frac{F^{\leftarrow}_{S_n^\xi}(q)}{H_n^{\leftarrow}(q)}, \frac{\mathbb{E}\Big(S_n^\xi - F^{\leftarrow}_{S_n^\xi}(q)\Big)^+}{\mathbb{E}\Big(S_n^\xi - H_n^{\leftarrow}(q)\Big)^+} \frac{\mathbb{E}\Big(S_n^\xi - H_n^{\leftarrow}(q)\Big)^+}{\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+}\right\}$$

$$\leqslant \frac{F^{\leftarrow}_{S_n^\xi}(q)}{H_n^{\leftarrow}(q)} \max\left\{1, \frac{\mathbb{E}\Big(S_n^\xi - H_n^{\leftarrow}(q)\Big)^+}{\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+} \sup_{u\geqslant H_n^{\leftarrow}(q)} \frac{\mathbb{P}\Big(S_n^\xi > u\frac{F^{\leftarrow}_{S_n^\xi}(q)}{H_n^{\leftarrow}(q)}\Big)}{\mathbb{P}\big(S_n^\xi > u\big)}\right\}, \tag{4.26}$$

because, according to the alternative expectation formula,

$$\frac{\mathbb{E}(\eta - x_1)^+}{\mathbb{E}(\eta - x_2)^+} = \frac{\int\limits_{x_1}^{\infty} \mathbb{P}(\eta > u)\mathrm{d}u}{\int\limits_{x_2}^{\infty} \mathbb{P}(\eta > u)\mathrm{d}u} \leqslant \frac{x_1}{x_2} \sup_{u\geqslant x_2} \frac{\mathbb{P}\big(\eta > u\frac{x_1}{x_2}\big)}{\mathbb{P}(\eta > u)}$$

for all positive $x_1, x_2$ and every r.v. $\eta$ such that $\mathbb{P}(\eta > x) > 0$, $x \in \mathbb{R}$.

Now let $\varepsilon \in (0, \frac{1}{2})$. By relations (4.25) and (4.26),

$$\frac{HG_q\Big(S_n^\xi\Big)}{H_n^{\leftarrow}(q) + \frac{1}{1-q} \sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+}$$

$$\leqslant (1+\varepsilon) \max\left\{1, \frac{\mathbb{E}\Big(S_n^\xi - H_n^{\leftarrow}(q)\Big)^+}{\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+} \sup_{u\geqslant H_n^{\leftarrow}(q)} \frac{\mathbb{P}\big(S_n^\xi > u(1-\varepsilon)\big)}{\mathbb{P}\big(S_n^\xi > u\big)}\right\}$$

for all $q$ sufficiently close to the unit from the left. Since $H_n^{\leftarrow}(q) \to \infty$ if $q \uparrow 1$, from the last estimate one can obtain

$$\limsup_{q\uparrow 1} \frac{HG_q\Big(S_n^\xi\Big)}{H_n^{\leftarrow}(q) + \frac{1}{1-q} \sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+} \leqslant (1+\varepsilon) \max\Bigg\{1,$$

$$\limsup_{q\uparrow 1} \frac{\mathbb{E}\Big(S_n^\xi - H_n^{\leftarrow}(q)\Big)^+}{\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+} \limsup_{q\uparrow 1} \sup_{u\geqslant H_n^{\leftarrow}(q)} \frac{\mathbb{P}\big(S_n^\xi > u(1-\varepsilon)\big)}{\mathbb{P}\big(S_n^\xi > u\big)}\Bigg\}$$

$$= (1+\varepsilon) \max \left\{ 1, \limsup_{x\to\infty} \frac{\mathbb{E}\Big(S_n^\xi - x\Big)^+}{\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - x\Big)^+} \limsup_{x\to\infty} \sup_{u\geqslant x} \frac{\mathbb{P}\big(S_n^\xi > u(1-\varepsilon)\big)}{\mathbb{P}\big(S_n^\xi > u\big)} \right\}$$

The fact that $F_{S_n^\xi} \in \mathscr{C}$, relation (4.2) of Theorem 4.1, and the arbitrariness of $\varepsilon \in (0, \frac{1}{2})$ imply that

$$\limsup_{q\uparrow 1} \frac{HG_q\Big(S_n^\xi\Big)}{H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+} \leqslant 1. \tag{4.27}$$

In a similar way, one can obtain

$$\liminf_{q\uparrow 1} \frac{HG_q\Big(S_n^\xi\Big)}{H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+}$$

$$\geqslant (1-\varepsilon) \min \left\{ 1, \liminf_{x\to\infty} \frac{\mathbb{E}\big(S_n^\xi - x\big)^+}{\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\big(\xi_k - x\big)^+} \liminf_{x\to\infty} \inf_{u\geqslant x} \frac{\mathbb{P}\big(S_n^\xi > u(1+\varepsilon)\big)}{\mathbb{P}\big(S_n^\xi > u\big)} \right\}$$

for all $\varepsilon \in (0, \frac{1}{2})$, and, under similar argumentations, one can derive

$$\liminf_{q\uparrow 1} \frac{HG_q\Big(S_n^\xi\Big)}{H_n^{\leftarrow}(q) + \frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\Big(\xi_k - H_n^{\leftarrow}(q)\Big)^+} \geqslant 1.$$

The last estimate, together with inequality (4.27), finishes the proof of the first part of Theorem 4.5.

Now suppose $\varkappa \geqslant 2$ and all conditions of part (ii) of Theorem 4.5 are satisfied. Due to Theorem 4.2,

$$HG_q\big(S_n^\xi\big) = \widetilde{x} + \left( \frac{\mathbb{E}\Big(\big(S_n^\xi - \widetilde{x}\big)^+\Big)^\varkappa}{1-q} \right)^{\frac{1}{\varkappa}}, \tag{4.28}$$

where $\widetilde{x} = \widetilde{x}(q)$ is the unique solution of the equation

$$\frac{\left(\mathbb{E}\Big(\big(S_n^\xi - \widetilde{x}\big)^+\Big)^{\varkappa-1}\right)^{\varkappa}}{\left(\mathbb{E}\Big(\big(S_n^\xi - \widetilde{x}\big)^+\Big)^{\varkappa}\right)^{\varkappa-1}} = \gamma_{S_n^\xi}(\widetilde{x}) = 1 - q.$$

Let $H_n$ be the d.f. defined in part (i) of Theorem 4.5. Due to asymptotic relation (4.23), upper bound in (4.25), and conditions of the theorem, d.f.s $F_{S_n^\xi}$ and $H_n$ both belong to the class $\mathscr{C}$. In addition, according to relation (4.23), the min-max inequality, and condition (4.6),

$$\begin{aligned}
&\left(\limsup_{x\to\infty} \frac{\overline{F}_{S_n^\xi}(yx)}{\overline{F}_{S_n^\xi}(x)}\right)^{\varkappa} \left(\limsup_{x\to\infty} \frac{\overline{F}_{S_n^\xi}(x)}{\overline{F}_{S_n^\xi}(yx)}\right)^{\varkappa-1} \\
&= \left(\limsup_{x\to\infty} \frac{\overline{H}_n(yx)}{\overline{H}_n(x)}\right)^{\varkappa} \left(\limsup_{x\to\infty} \frac{\overline{H}_n(x)}{\overline{H}_n(yx)}\right)^{\varkappa-1} \\
&\leqslant \left(\max_{k\in\mathcal{I}_n} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(yx)}{\overline{F}_{\xi_k}(x)}\right)^{\varkappa} \left(\max_{k\in\mathcal{I}_n} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_k}(yx)}\right)^{\varkappa-1} < 1 \qquad (4.29)
\end{aligned}$$

for some $y > 1$. Therefore, Remark 4.2 implies that the d.f. $1 - \gamma_{S_n^\xi}$ belongs to the class $\mathscr{C} \cap \mathcal{PD}$.

For sufficiently large $x$, by the alternative expectation formula, it holds that

$$\begin{aligned}
\frac{\left(\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\big((\xi_k - x)^+\big)^{\varkappa-1}\right)^{\varkappa}}{\left(\sum\limits_{k\in\mathcal{I}_n} \mathbb{E}\big((\xi_k - x)^+\big)^{\varkappa}\right)^{\varkappa-1}} &= \frac{\left((\varkappa-1)\int\limits_x^\infty (u-x)^{\varkappa-2}\overline{H}_n(u)\mathrm{d}u\right)^{\varkappa}}{\left(\varkappa\int\limits_x^\infty (u-x)^{\varkappa-1}\overline{H}_n(u)\mathrm{d}u\right)^{\varkappa-1}} \\
&= \frac{\left(\mathbb{E}\big((\zeta_n - x)^+\big)^{\varkappa-1}\right)^{\varkappa}}{\left(\mathbb{E}\big((\zeta_n - x)^+\big)^{\varkappa}\right)^{\varkappa-1}} := \gamma_{\zeta_n}(x),
\end{aligned}$$

where $\zeta_n$ is a r.v. with d.f. $H_n$. Due to estimate (4.29), the d.f. $F_{\zeta_n} = H_n$ satisfies condition (4.19) which, according to Remark 4.2, implies that the d.f. $1-\gamma_{\zeta_n}$ belongs to the class $\mathscr{C}\cap\mathcal{PD}$ together with the d.f. $1-\gamma_{S_n^\xi}$.

On the other hand, relation (4.2) of Theorem 4.1 implies that

$$\gamma_{S_n^\xi}(x) \sim \gamma_{\zeta_n}(x).$$

Consequently, all conditions of Lemma 4.4 are satisfied with $c = 1$ and d.f.s $1 - \gamma_{S_n^\xi}$, $1 - \gamma_{\zeta_n}$. According to this lemma,

$$\widetilde{x}(q) \underset{q\uparrow 1}{\sim} \widehat{x}(q). \tag{4.30}$$

Now the proof of this part continues in the same way as in the proof of part (i). For all $q \in (0,1)$, by relation (4.28) and the min-max inequality,

$$\frac{HG_q\big(S_n^\xi\big)}{\widehat{x}(q) + \left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{\frac{1}{\varkappa}}}$$

$$\leqslant \max\left\{\frac{\widetilde{x}(q)}{\widehat{x}(q)}, \left(\frac{\mathbb{E}\Big(\big(S_n^\xi - \widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}(q)\big)^+\Big)^{\varkappa}}\right)^{\frac{1}{\varkappa}}\right\}$$

$$= \max\left\{\frac{\widetilde{x}(q)}{\widehat{x}(q)}, \left(\frac{\mathbb{E}\Big(\big(S_n^\xi - \widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widetilde{x}(q)\big)^+\Big)^{\varkappa}} \frac{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}(q)\big)^+\Big)^{\varkappa}}\right)^{\frac{1}{\varkappa}}\right\}.$$

If $q$ is sufficiently close to the unit from the left, then

$$\frac{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k - \widehat{x}(q)\big)^+\Big)^{\varkappa}} = \frac{\int\limits_{\widetilde{x}(q)}^{\infty}(u - \widetilde{x}(q))^{\varkappa-1}\overline{H}_n(u)\mathrm{d}u}{\int\limits_{\widehat{x}(q)}^{\infty}(u - \widehat{x}(q))^{\varkappa-1}\overline{H}_n(u)\mathrm{d}u}$$

$$= \left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}\right)^{\varkappa} \frac{\int\limits_{\widehat{x}(q)}^{\infty}(u - \widehat{x}(q))^{\varkappa-1}\overline{H}_n\left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}u\right)\mathrm{d}u}{\int\limits_{\widehat{x}(q)}^{\infty}(u - \widehat{x}(q))^{\varkappa-1}\overline{H}_n(u)\mathrm{d}u}$$

$$\leqslant \left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}\right)^{\varkappa} \sup_{u\geqslant\widehat{x}(q)} \frac{\overline{H}_n\left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}u\right)}{\overline{H}_n(u)}.$$

Consequently, for $q$ sufficiently close to the unit from the left,

$$\frac{HG_q\big(S_n^{\xi}\big)}{\widehat{x}(q)+\left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{\frac{1}{\varkappa}}}$$

$$\leqslant \frac{\widetilde{x}(q)}{\widehat{x}(q)}\max\left\{1,\left(\frac{\mathbb{E}\Big(\big(S_n^{\xi}-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}{\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widetilde{x}(q)\big)^+\Big)^{\varkappa}}\sup_{u\geqslant\widehat{x}(q)}\frac{\overline{H}_n\left(\frac{\widetilde{x}(q)}{\widehat{x}(q)}u\right)}{\overline{H}_n(u)}\right)^{\frac{1}{\varkappa}}\right\}.$$

Since $\widetilde{x}(q)\underset{q\uparrow 1}{\to}\infty$, $\widetilde{x}(q)\underset{q\uparrow 1}{\sim}\widehat{x}(q)$, and the d.f. $H_n$ belongs to the class $\mathscr{C}$, the last estimate and asymptotic relation (4.2) of Theorem 4.1 imply that

$$\limsup_{q\uparrow 1}\frac{HG_q\big(S_n^{\xi}\big)}{\widehat{x}(q)+\left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{\frac{1}{\varkappa}}}\leqslant 1.$$

Similarly, one can derive that

$$\liminf_{q\uparrow 1}\frac{HG_q\big(S_n^{\xi}\big)}{\widehat{x}(q)+\left(\frac{1}{1-q}\sum\limits_{k\in\mathcal{I}_n}\mathbb{E}\Big(\big(\xi_k-\widehat{x}(q)\big)^+\Big)^{\varkappa}\right)^{\frac{1}{\varkappa}}}\geqslant 1.$$

This finishes the proof of the second part of Theorem 4.5. $\square$

### 4.3.3 Proof of Theorem 4.6

Consider the case $\varkappa=1$. According to the conditions and Lemma 4.10, a collection of r.v.s $\{\theta_1X_1,\ldots,\theta_nX_n\}$ follows the pQAI dependence structure. By Lemma 4.11, d.f.s $F_{\theta_kX_k}$ belong to the class $\mathscr{C}$ for all $k\in\{1,2,\ldots,n\}$. Since $\max\limits_{1\leqslant k\leqslant n}\mathbb{E}X_k^+<\infty$ and $p>\max\limits_{1\leqslant k\leqslant n}J^+_{F_{X_k}}$, by Lemma 3.5 of [102], $\mathbb{E}\theta_kX_x^+<\infty$ for all $k\in\{1,2,\ldots,n\}$, because

$$\sup\left\{v:\int\limits_{[0,\infty)}x^v\mathrm{d}F(x)<\infty\right\}\leqslant J_F^+$$

for every d.f. $F$ from the class $\mathscr{D}$. Finally, by Lemma 4.12, $F_{\theta_k X_k} \in \mathcal{PD}$ for all $k \in \{1, 2, \ldots, n\}$. Hence the collection of r.v.s $\{\theta_1 X_1, \ldots, \theta_n X_n\}$ satisfies all conditions of part (i) of Theorem 4.5. This implies the first statement of Theorem 4.6.

Now, consider the second part of the theorem. If $\varkappa \geqslant 2$, then the second part of Theorem 4.6 follows from the second part of Theorem 4.5 by completely analogous reasoning as in the first part of the proof, only instead of Lemma 4.12, one needs to use Lemma 4.13. $\square$

## 4.4 Example

This section presents an illustrative example demonstrating the application of Theorem 4.5 to the evaluation of the HG risk measure when the risk level is close to unity and the leading risk has a consistently varying distribution.

Consider a collection of independent r.v.s $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ with the following d.f.s:

$$F_1(x) = \left(1 - \frac{1}{(x+1)^3}\right) \mathbb{1}_{[0,\infty)}(x),\ F_2(x) = \left(1 - \frac{1}{2(x+1)^3}\right) \mathbb{1}_{[0,\infty)}(x),$$

$$F_3(x) = F_4(x) = \left(1 - \mathrm{e}^{-x}\right) \mathbb{1}_{[0,\infty)}(x).$$

The analysis focuses on the derivation of asymptotic formulas for the $HG_q$ risk measure corresponding to power Young functions of the form $\varphi(t) = t^{\varkappa}$, with $\varkappa = 1$ and $\varkappa = 2$.

It is clear that the above collection of r.v.s satisfies the conditions of Theorem 4.5 with $\mathcal{I}_4 = \{1, 2\}$.

In the first case, when $\varkappa = 1$,

$$HG_q\big(S_4^{\xi}\big) \underset{q\uparrow 1}{\sim} H_4^{\leftarrow}(q) + \frac{1}{1-q}\left(\mathbb{E}\big(\xi_1 - H_4^{\leftarrow}(q)\big)^+ + \mathbb{E}\big(\xi_2 - H_4^{\leftarrow}(q)\big)^+\right),$$

where $H_4^{\leftarrow}$ is the quantile function of the d.f.

$$H_4 = \max\left\{0, 1 - \overline{F}_1 - \overline{F}_2\right\}.$$

For large $x$,

$$H_4(x) = 1 - \frac{3}{2}\frac{1}{(1+x)^3}.$$

Therefore,

$$H_4^{\leftarrow}(q) = \frac{1}{\sqrt[3]{\frac{2}{3}(1-q)}} - 1$$

for $q$ sufficiently close to unity. For all $x \geqslant 0$,

$$\mathbb{E}(\xi_1 - x)^+ = \frac{1}{2(x+1)^2}, \quad \mathbb{E}(\xi_2 - x)^+ = \frac{1}{4(x+1)^2}.$$

Therefore, in this case,

$$HG_q\big(S_4^{\xi}\big) \underset{q\uparrow 1}{\sim} \frac{1}{\sqrt[3]{\frac{2}{3}(1-q)}} - 1 + \frac{1}{1-q}\frac{3}{4}\left(\sqrt[3]{\frac{2}{3}(1-q)}\right)^2$$

$$= \frac{1}{\sqrt[3]{1-q}}\left(\frac{3}{2}\right)^{\frac{4}{3}} - 1.$$

Figure 4.1 displays the function defined above together with Monte Carlo estimates of $HG_q\big(S_4^{\xi}\big)$.

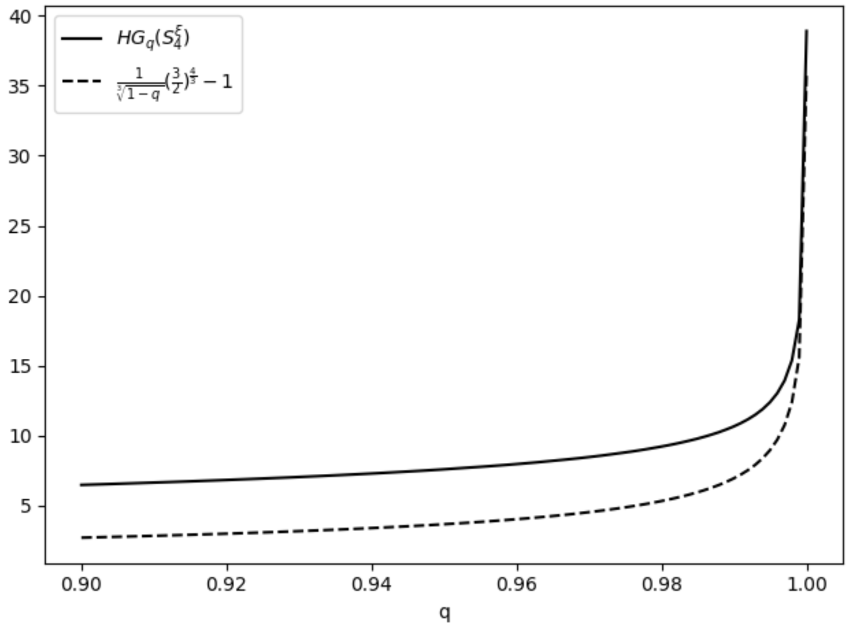


**FIGURE 4.1.** Simulated and asymptotically approximated values of $HG_q\big(S_4^{\xi}\big)$, where $\varkappa = 1$

Consider the second case, where $\varkappa = 2$. Then,

$$HG_q\big(S_4^\xi\big) \underset{q\uparrow 1}{\sim} \widehat{x}(q) + \frac{1}{\sqrt{1-q}}\Big(\mathbb{E}\big((\xi_1 - \widehat{x}(q))^+\big)^2 + \mathbb{E}\big((\xi_2 - \widehat{x}(q))^+\big)^2\Big)^{1/2},$$

where $\widehat{x}(q)$ is the solution of the equation

$$\frac{\big(\mathbb{E}(\xi_1 - x)^+ + \mathbb{E}(\xi_2 - x)^+\big)^2}{\mathbb{E}\big((\xi_1 - x)^+\big)^2 + \mathbb{E}\big((\xi_2 - x)^+\big)^2} = 1 - q.$$

In this case,

$$\mathbb{E}(\xi_1 - x)^+ = \frac{1}{2(x+1)^2}, \quad \mathbb{E}(\xi_2 - x)^+ = \frac{1}{4(x+1)^2},$$

$$\mathbb{E}\big((\xi_1 - x)^+\big)^2 = \frac{1}{x+1}, \quad \mathbb{E}\big((\xi_2 - x)^+\big)^2 = \frac{1}{2(x+1)}.$$

Therefore,

$$\widehat{x}(q) = \sqrt[3]{\frac{3}{8(1-q)}} - 1$$

and

$$HG_q\big(S_4^\xi\big) \underset{q\uparrow 1}{\sim} \frac{3}{2}\sqrt[3]{\frac{3}{1-q}} - 1.$$

Figure 4.2 displays the function defined above together with Monte Carlo estimates of $HG_q\big(S_4^\xi\big)$.

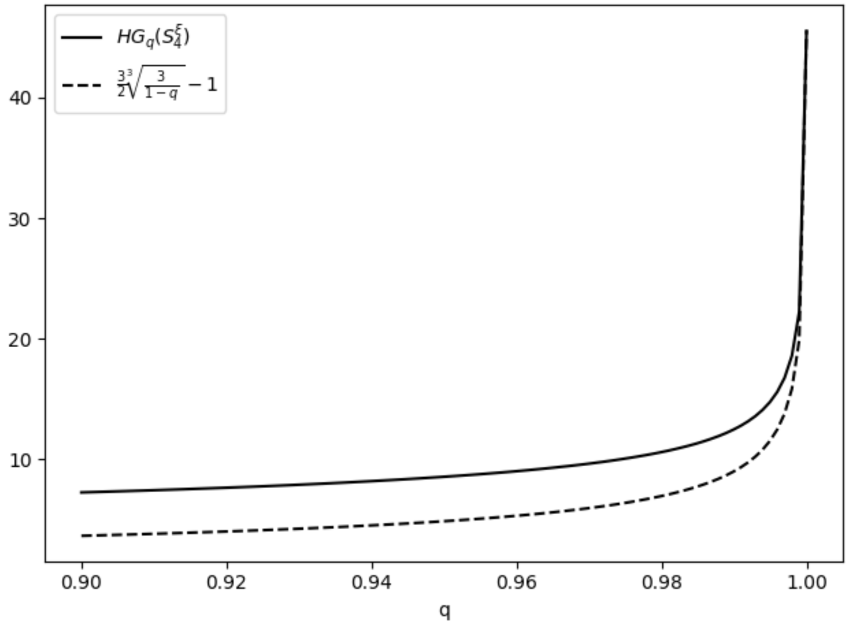


**FIGURE 4.2.** Simulated and asymptotically approximated values of $HG_q\left(S_4^\xi\right)$, where $\varkappa = 2$

## CONCLUSIONS

A summary of the main results obtained in this dissertation is presented below.

Chapter 2 extends Theorems 3 and 4 of [33] by allowing a broader class of transformations beyond degree functions. It is shown that the following asymptotic bounds for generalized left truncated moments of sums with pQAI dominatedly varying summands hold

$$\sum_{k=1}^{n} L_{\xi_k}\mathbb{E}\left(\varphi(\xi_k)\mathbb{1}_{\{\xi_k>x\}}\right) \lesssim \mathbb{E}\left(\varphi(S_n^{\xi})\mathbb{1}_{\{S_n^{\xi}>x\}}\right)$$

$$\lesssim \sum_{k=1}^{n} \frac{1}{L_{\xi_k}}\mathbb{E}\left(\varphi(\xi_k)\mathbb{1}_{\{\xi_k>x\}}\right)$$

and, analogously, for randomly weighted sums,

$$\sum_{k=1}^{n} L_{\xi_k}\mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{1}_{\{\theta_k\xi_k>x\}}\right) \lesssim \mathbb{E}\left(\varphi(S_n^{\theta\xi})\mathbb{1}_{\{S_n^{\theta\xi}>x\}}\right)$$

$$\lesssim \sum_{k=1}^{n} \frac{1}{L_{\xi_k}}\mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{1}_{\{\theta_k\xi_k>x\}}\right).$$

under transformation $\varphi : \mathbb{R} \to \mathbb{R}_+ = [0, \infty)$, satisfying the following properties

(i) $\varphi$ is eventually nondecreasing;

(ii) $\varphi$ is eventually subhomogeneous;

(iii) $\varphi$ is eventually differentiable;

(iv) for each $k \in \{1, \ldots, n\}$, the generalized moment $\mathbb{E}(\varphi(\xi_k)\mathbb{1}_{\{\xi_k>x\}})$ (or, respectively, $\mathbb{E}\left(\varphi(\theta_k\xi_k)\mathbb{1}_{\{\theta_k\xi_k>x\}}\right)$) is eventually finite.

Chapter 3 investigates randomly stopped sums $S_\nu$ and shows how the membership of the distribution of $S_\nu$ in the classes $\mathscr{ERV}$ or $\mathscr{D}$ influences the membership of the counting r.v. $\nu$ distribution in the class $\mathscr{D}$. These results extend Proposition 4.9 of [39], which was originally formulated for the more restrictive class $\mathscr{R}$. Moreover, asymptotic formulas relating the tail of a randomly stopped sum from class $\mathscr{ERV}_{\{\alpha,\beta\}}$ to the tail of a counting random variable were derived:

$$(\mathbb{E}\xi)^{\alpha}\overline{F}_{\nu}(x) \lesssim \overline{F}_{S_{\nu}}(x) \lesssim (\mathbb{E}\xi)^{\beta}\overline{F}_{\nu}(x) \quad \text{if} \quad \mathbb{E}\xi > 1$$

$$(\mathbb{E}\xi)^{\beta}\overline{F}_{\nu}(x) \lesssim \overline{F}_{S_{\nu}}(x) \lesssim (\mathbb{E}\xi)^{\alpha}\overline{F}_{\nu}(x) \quad \text{if} \quad \mathbb{E}\xi < 1$$

$$\overline{F}_{S_{\nu}}(x) \sim \overline{F}_{\nu}(x) \quad \text{if} \quad \mathbb{E}\xi = 1$$

Finally, assuming that r.v.s $\xi_1, \ldots, \xi_n$ represent individual random losses, Chapter 4 investigates properties of the Haezendonck-Goovaerts risk measure and extends results of [105]. It is shown that, in order to derive the asymptotics of the HG risk measure for the sum of random risks, it is sufficient to determine the appropriate characteristics of the sum of dominant risks. It was noticed that such result holds under the assumption that d.f.s of the dominant risks are consistently varying and satisfy additional requirements related to positively decreasing tails. Moreover, this finding implies that the asymptotic behaviour of the HG risk measure can be determined without computing complex convolutions. Instead, it is sufficient to construct the d.f. from the sum of d.f.s of the selected group. As a consequence to this result, an additional theorem is formulated, demonstrating that the procedure for determining the asymptotics of the HG risk measure also applies in cases where a set of r.v.s with random weights is considered.

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# SUMMARY IN LITHUANIAN

## Tyrimo objektas

Matematinis tyrimas, pateiktas šioje daktaro disertacijoje, yra skirtas sumų, kurių dėmenis sudaro sunkiauodegiai atsitiktiniai dydžiai, asimptotinei analizei.

Tiksliau tariant, dėmesys skiriamas kelioms atsitiktinių sumų klasėms. Pirmiausia nagrinėjamos atsitiktinės sumos su fiksuotu dėmenų skaičiumi

$$S_n^{\xi} = \xi_1 + ... + \xi_n$$

kartu su atsitiktinėmis svorinėmis sumomis

$$S_n^{\theta\xi} = \theta_1\xi_1 + ... + \theta_n\xi_n.$$

Antra, tiriamos atsitiktinai sustabdytos sumos

$$S_\nu := \begin{cases} 0 & \text{if } \nu = 0, \\ \xi_1 + \ldots + \xi_\nu & \text{if } \nu \geqslant 1, \end{cases}$$

kuriose dėmenų skaičius $\nu$ yra atsitiktinis dydis.

Visoms šioje disertacijoje nagrinėjamoms sumoms būdinga bendra savybė, kad jų dėmenys arba stabdymo momentas yra sunkiauodegiai atsitiktiniai dydžiai. Sunkias uodegas turintys skirstiniai pasižymi lėtu uodegos tikimybių nykimu ir atlieka svarbų vaidmenį modeliuojant ekstremalius įvykius. Tokie įvykiai yra neretai nagrinėjami draudimo ir finansų srityse, kai bendri portfelio nuostoliai yra nulemti individualių ekstremalių nuostolių. Šiame kontekste, klasikiniai analizės metodai, paremti eksponentiniais momentais ar momentus generuojančiomis funkcijomis, yra netaikytini, todėl naudojami alternatyvūs asimptotiniais metodai.

Disertacijoje taip pat nagrinėjamas agreguotų nuostolių rizikos vertinimas. Interpretuojant dėmenis kaip individualius atsitiktinius nuostolius, ypatingas dėmesys skiriamas Haezendonck–Goovaerts rizikos matui – suderintam bei apibrėžtam Orlicz'o erdvėje. Šis matas suteikia lankstų pagrindą rizikos valdymui ir uodegos jautrumo modeliavimui, leidžiantį kontroliuoti ekstremalių nuostolių įtaką pasirenkant atitinkamą Young funkciją. Disertacijoje analizuojama Haezendonck–Goovaerts rizikos mato asimptotinė elgsena, taikoma sunkiąsias uodegas turinčių rizikų sumoms, ir tiriama, kaip dominuojantys sumos elementai lemia šią elgseną.

## Tikslai ir uždaviniai

Šioje daktaro disertacijoje suformuoti trys pagrindiniai tyrimo tikslai, kurie šiame skyriuje trumpai aprašyti.

Pirma, nagrinėjama atsitiktinių ir atsitiktinių svorinių sumų su fiksuotu dėmenų skaičiumi apibendrintų nupjautinių momentų

$$\mathbb{E}(\varphi(S_n^{\xi})\mathbb{1}_{\{S_n^{\xi}>x\}}) \quad \text{ir} \quad \mathbb{E}(\varphi(S_n^{\theta\xi})\mathbb{1}_{\{S_n^{\theta\xi}>x\}})$$

asimptotinė elgsena, kai $x \to \infty$. Čia $\varphi : \mathbb{R} \to \mathbb{R}_+ = [0, \infty)$ žymi transformacijos funkciją, tenkinančią tam tikras sąlygas. Vienas iš pagrindinių disertacijos tikslų yra gauti šių apibendrintų nupjautinių momentų asimptotinius apatinį ir viršutinį rėžius, išreikštus individualių dėmenų nupjautinių momentų $\mathbb{E}(\varphi(\xi_k)\mathbb{1}_{\{\xi_k>x\}})$ arba, atitinkamai, $\mathbb{E}(\varphi(\theta_k\xi_k)\mathbb{1}_{\{\theta_k\xi_k>x\}})$ suma tam tikrų koreguojančių konstantų tikslumu. Tokio tipo uždaviniai jau buvo nagrinėti mokslinėje literatūroje. Pavyzdžiui, M. Dirma, S. Paukštys ir J. Šiaulys [33] nustatė asimptotinius apatinį ir viršutinį rėžius tuo atveju, kai transformacijos funkcija $\varphi$ yra laipsninė funkcija. Šios disertacijos tikslas – toliau apibendrinti šiuos rezultatus, leidžiant platesnę transformacijų klasę.

Antra, tyrimas papildomas atsitiktinai sustabdytų sumų $S_\nu$ analize. Tiksliau tariant, analizuojami atvirkštinio tipo uždaviniai, kai reguliarumo sąlygos yra duotos sumos $S_\nu$ pasiskistymui. Disertacijoje sprendžiamas klausimas, kaip skirstinio $S_\nu$ priklausymas tam tikroms sunkiauodegių skirstinių klasėms lemia stabdymo momento $\nu$ uodegos elgseną. G. Faÿ, B. González-Arévalo, T. Mikosch, G. Samorodnitsky [39] nagrinėjo atvejį, kai atsitiktinai sustabdyta suma $S_\nu$ priklauso reguliariai kintančių skirstinių klasei $\mathscr{R}_\alpha$ su indeksu $\alpha > 0$, ir nustatė sąlygas, kurioms esant stabdymo momentas $\nu$ taip pat priklauso tai pačiai klasei. Šios disertacijos tikslas – išplėsti šiuos rezultatus, parodant, kaip atsitiktinai sustabdytos sumos $S_\nu$ reguliarumas, t.y. priklausymas platesnėms skirstinių klasėms, tokioms kaip išplėstinio reguliaraus kintamumo $\mathscr{ERV}$ ar dominuojančio kintamumo $\mathscr{D}$ klasėms, lemia stabdymo momento $\nu$ reguliarumą dominuojančio kintamumo skirstinių klasėje $\mathscr{D}$.

Galiausiai nagrinėjamas Haezendonck–Goovaerts rizikos mato, taikomo sunkiauodegių rizikų sumoms, asimptotinis elgesys. Disertacijos tikslas yra nustatyti sąlygas, kurioms esant Haezendonck–Goovaerts rizikos mato asimptotika yra nulemta dominuojančių sumos dėmenų, ir išvesti asimptotines formules, leidžiančias išvengti sudėtingų konvoliucijos skaičiavimų. Panašus uždavinys jau buvo nagrinėtas mokslinėje literatūroje. Pavyzdžiui, Tang ir Yang [105] išvedė Haezendonck–Goovaerts rizikos mato asimptotinę išraišką tuo atveju, kai Young funkcija yra laipsninė funkcija $\varphi(t) = t^{\varkappa}$ su $\varkappa \geqslant 1$, o nagrinėjama atsitiktinė rizika priklauso reguliariai kintančių skirstinių klasei $\mathscr{R}_\alpha$ su indeksu $\alpha > \varkappa$. Šios disertacijos tikslas – išplėsti šiuos rezultatus nagrinėjant atsitiktinių rizikų sumas, kurių kai kurie arba visi dėmenys priklauso platesniam sunkiauodegų skirstinių poklasiui, t.y. nuosaikiai kintančių skirstinių klasei $\mathscr{C}$. Be to, analizė išplečiama ir atsitiktinių svorinių sumų atvejui.

## Aktualumas ir praktinė motyvacija

Nors ši daktaro disertacija yra teorinio pobūdžio, atsitiktinių sumų aktualumas išlieka praktiniuose taikymuose. Šios sumos natūraliai atsiranda uždaviniuose, susijusiuose su draudimo ir finansų rizikos valdymu.

Sumos su fiksuotu dėmenų skaičiumi dažnai pasitaiko praktiniuose modeliavimo uždaviniuose draudimo kontekste. Diskretaus laiko rizikos modeliai yra vienas iš šių pavyzdžių. Šiuose modeliuose atsitiktiniai dydžiai $\xi_k$ gali atitikti draudimo bendrovės grynuosius nuostolius laikotarpiais $(k-1, k]$, apibrėžtus kaip bendros išmokėtų žalų sumos ir bendrų draudimo įmokų pajamų skirtumas per laikotarpį $(k-1, k]$. Atitinkami atsitiktiniai svoriai $\theta_k$ gali būti interpretuojami kaip stochastiniai diskonto faktoriai, transformuojantys būsimus nuostolius momentu $k$ į jų dabartinę vertę momentu 0. Pagal šią interpretaciją atsitiktinė svorinė suma $S_n^{\theta\xi}$ aprašo bendrovės bendrą grynųjų nuostolių dabartinę vertę baigtiniame laiko intervale $(0, n]$.

Kita svarbi su draudimu susijusi taikymo sritis yra individualios rizikos modeliai. Šiuose modeliuose laikoma, kad draudimo bendrovė valdo portfelį, sudarytą iš $n$ individualių draudimo sutarčių. Tuomet suma $S_n^{\theta\xi}$

interpretuojama kaip bendra žalų suma, patirta viso draudimo portfelio, o kiekvienas narys $\theta_k\xi_k$, $k = 1, \ldots, n$, atitinka žalą, susijusią su $k$-ąja draudimo sutartimi. Kadangi egzistuoja galimybė, jog tam tikrai sutarčiai žala nebus patirta, atsitiktinis svoris $\theta_k$ dažnai modeliuojamas indikatoriumi (Bernulio atsitiktiniu dydžiu), aprašančiu $k$-osios žalos įvykį, kai $\theta_k = 1$ reiškia, jog žala įvyko, o $\theta_k = 0$ kitais atvejais. Atsitiktinis dydis $\xi_k$ tuomet aprašo $k$-osios sutarties žalos dydį su sąlyga, kad žala įvyko.

Atsitiktinės svorinės sumos taip pat atlieka svarbų vaidmenį finansų rizikos valdyme, ypač portfelių modeliavime ir kapitalo paskirstyme. Finansinis portfelis gali būti laikomas kaip rinkinys $n$ atskirų rizikos komponentų, pavyzdžiui, turto klasių ar verslo linijų. Šiame kontekste atsitiktinis dydis $\xi_k$ atitinka nuostolį, susijusį su $k$-uoju komponentu. Priklausomai nuo modeliavimo perspektyvos, $\theta_k$ gali būti interpretuojamas kaip stochastinis diskonto faktorius, taikomas būsimiems nuostoliams, arba kaip atsitiktinis $k$-ojo portfelio komponento svoris. Atitinkamai, suma $S_n^{\theta\xi}$ gali būti interpretuojama arba kaip viso portfelio nuostolių dabartinė vertė, arba kaip agreguotas svertinis portfelio nuostolis. Abiem atvejais atskirų komponentų priklausymas sunkiauodegiems skirstiniams gali lemti situacijas, kai nedidelis skaičius dominuojančių rizikų iš esmės nulemia bendrą portfelio riziką.

Įvairiuose praktiniuose taikymuose taip pat pasitaiko atsitiktinai sustabdytos sumos. Šios sumos atsiranda tuomet, kai dėmenų skaičius yra nežinomas, t.y. atsitiktinis. Draudimo ir finansų srityse tai gali būti susiję su atsitiktine sutarties trukme ar atsitiktiniu žalų skaičiumi. Išgyvenamumo analizėje atsitiktinai sustabdytos sumos naudojamos modeliuoti agreguotas reikšmes, sukauptas iki atsitiktinio veiklos nutraukimo momento.

Galiausiai, šios disertacijos praktinė reikšmė glaudžiai susijusi su rizikos vertinimu. Interpretuojant sumos narius kaip individualius atsitiktinius nuostolius, Haezendonck–Goovaerts rizikos matas suteikia lankstų ir teoriškai pagrįstą pagrindą agreguotos rizikos kiekybiniam vertinimui. Šio rizikos mato formulavimas Orlicz erdvėse leidžia kontroliuoti jautrumą ekstremaliems nuostoliams, pasirenkant atitinkamą Young funkciją. Praktinis šio rizikos mato taikymas dažnai apsunkinamas sudėtingų skaičiavimų. Šioje disertacijoje gauti asimptotiniai rezultatai parodo, kad, esant tinkamoms sąlygoms, Haezendonck–Goovaerts rizikos mato elgsena agreguotiems nuostoliams yra nulemta dominuojančių sumos dėmenų. Ši įžvalga leidžia reikšmingai supaprastinti rizikos vertinimą ir suteikia teoriškai pagrįstą pagrindą sudėtingų rizikos matų aproksimacijai praktiniame kontekste.

## Disertacijos struktūra

Šiame skyriuje pateikiama daktaro disertacijos struktūra.

Darbas parengtas anglų kalba ir prasideda trumpa įžanga.

1 skyriuje pateikiami disertacijoje naudojami žymėjimai ir apibrėžimai, sudarantys pagrindą tolimesnei analizei. Apžvelgiamios sunkiauodegių skirstinių klasės ir jų poklasiai; aprašomos priklausomumo struktūros, dažnai sutinkamos kartu su sunkiauodegiais atsitiktiniais dydžiais; primenamos pagrindinės rizikos matų sąvokos ir pateikiamas Haezendonck–Goovaerts rizikos mato apibrėžimas.

2 skyrius skirtas sumoms su fiksuotu dėmenų skaičiumi. Jame nagrinėjama apibendrintų nupjautinių momentų asimptotinė elgsena tiek nesvorinių, tiek svorinių atsitiktinių sumų atveju, esant įvairioms prielaidoms apie sumos narių uodegų elgseną ir transformacijos funkciją. Pagrindiniai šio skyriaus rezultatai pateikia apibendrintų nupjautinių momentų asimptotinius apatinį ir viršutinį rėžius agreguotoms sumoms, išreikštus atitinkamais individualių sumos dėmenų momentais. Šie rezultatai taip pat publikuoti [32].

3 skyrius skirtas atsitiktinai sustabdytų sumų analizei. Šiame skyriuje nagrinėjami atvirkštinio tipo uždaviniai, susiję su atsitiktinai sustabdytos sumos uodegos elgsenos ir stabdymo momento sąryšiu. Tiksliau tariant, nustatomos sąlygos, kurioms esant atsitiktinai sustabdytos sumos priklausymas tam tikram sunkiauodegių skirstinių poklasiui lemia specifines sustabdymo momento uodegos savybes. Šie rezultatai grindžiami publikacija [91].

4 skyrius nagrinėja Haezendonck–Goovaerts rizikos mato, taikomo sunkiauodegių rizikų sumoms, asimptotinę elgseną. Parodoma, kad, esant tinkamoms sąlygoms, rizikos mato asimptotika yra nulemta dominuojančių sumos komponentų, kas leidžia reikšmingai supaprastinti mato

skaičiavimą. Šiame skyriuje gauti rezultatai taip pat gali būti taikomi atsitiktinėms svorinėms sumoms. Šie rezultatai buvo publikuoti [92].

2, 3 ir 4 skyriai pasižymi panašia struktūra. Kiekvienas skyrius pradedamas trumpa literatūros apžvalga, po to pateikiami pagrindiniai rezultatai ir jų įrodymai, o skyrius baigiamas bent vienu iliustraciniu pavyzdžiu, demonstruojančiu gautų rezultatų taikomumą. Visi šiuose skyriuose pateikti pavyzdžiai buvo sugeneruoti naudojant „Python“ programavimo kalbą.

Be pagrindinių skyrių, disertacijoje taip pat pateikiama pagrindinių rezultatų santrauka bei literatūros sąrašas.

Šioje disertacijoje pateikti rezultatai buvo gauti autorės, įskaitant bendrus mokslinius straipsnius su kitais bendraautoriais, kurių sąrašas pateiktas DISSEMINATION OF RESULTS skyriuje.

Galiausiai, disertacijos pabaigoje pateikiamas autorės gyvenimo aprašymas.

# DISSEMINATION OF RESULTS

The main results of this thesis have been published in the following scientific papers:

1. M. Dirma, N. Nakliuda, J. Šiaulys, Generalized Moments of Sums with Heavy-Tailed Random Summands, *Lithuanian Mathematical Journal*, **63**(3):254-271, 2023. DOI: https://doi.org/10.1007/s10986-023-09606-y

2. J. Šiaulys, A. Elijio, R. Leipus, N. Nakliuda, Regularity of a Randomly Stopped Sum Determines Regularity of the Stopping Moment, *Lithuanian Mathematical Journal*, **65**(3):380–403, 2025. DOI: https://doi.org/10.1007/s10986-025-09685-z

3. J. Šiaulys, M. Dirma, N. Nakliuda, L. Zanardelli, Asymptotic Formulas for the Haezendonck-Goovaerts Risk Measure of Sums with Consistently Varying Increaments, *Axioms*, **15**(1):20, 2026. DOI: https://doi.org/10.3390/axioms15010020

The results obtained during the preparation of this thesis were presented at the following conferences and seminars:

1. M. Dirma, N. Nakliuda, *Asimptotinės formulės sunkiauodegių skirstinių sumų apibendrintiems momentams*, Finansų ir draudimo matematikos seminaras, 13 December 2022, Vilnius

2. M. Dirma, N. Nakliuda, *Haezendonck-Goovaerts rizikos mato asimptotika skirstinių su nuosaikiai kintančiomis uodegomis sumoms*, Lietuvos matematikų draugijos 66-oji konferencija, 26 June 2025, Klaipėda

3. N. Nakliuda, *Generalized Moments of Sums with Heavy Tailed Random Summands*, 1st International Vilnius Conference on Statistics and Its Applications, 28 August 2025, Vilnius

4. N. Nakliuda, *Asymptotic formulas for the Haezendonck-Goovaerts risk measure of sums with consistently varying increaments*, International Conference on Probability Theory and Number Theory, 15 September 2025, Palanga

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